Appendix A

Primer on Action

A.1 Newtonian Dynamics

Dynamics is the study of the causes of motion. The motion is the temporal evolution of systems in space. Newtonian physics is based on the idea that space and time are absolute. They are unaffected by what is in it and how it moves. The principles of dynamics were first fully articulated by Issac Newton in his Principia, [Cohen & Whitman 1999]

A primary notion is that there are forces. These forces represent the effect of other bodies on the body whose motion is under study. Your third-grade definition of a force–a push or a pull–is as good as any for a start. In this sense, forces are contact actions of one body on another. To do physics, we need to expand this idea beyond contact forces to action at a distance influences, see Section 2.1. To get a better understanding of forces, consider the world made up of several parts. This system of parts is isolated and thus all influences are from the parts on each other. This is the essence of reductionism: you can reduce the whole to its parts and the effect of any part on a given part does not depend on the remaining other parts. The important point is that a force is the effect of one body on another and is only considered when you replace the body by its force, see Figure A.1. For instance, if we are interested in the motion of body one. We talk about the force of body two on body one and the force of body three on body one and so forth. Once we know the forces and use the fact that force in simple cases is a vector quantity and obeys the usual rules for vector addition, we can get the total force by addition. In a very real sense in the analysis of the motion of body one, bodies two and three etc. are replaced by their forces. With the advent of a fundamental field theory initiated by Maxwell, see Section 4, we have to broaden our idea of force so that it becomes separated from the body that is its source and just talk about it as a thing unto itself.

For now, all forces are due to other bodies and they have meaning only in the sense that they are there when we want to discuss the effect that one body has on the other.

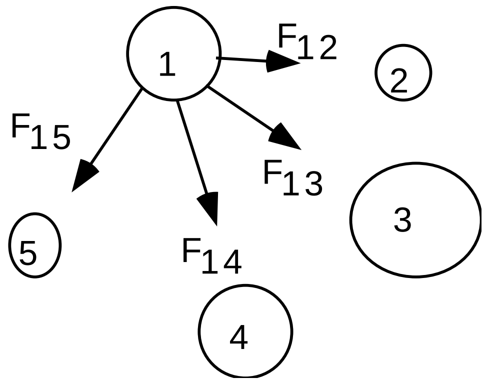

Figure A.1: **Adding Forces** A system composed of 5 parts. The forces are there in the sense that F_{12} is the push or pull on body 1 due to body 2. F_{12} can depend only on the relationship between bodies 1 and 2 and F_{12} does not depend on the presence of the other bodies. Similarly F_{1i} is the effect of body i on body 1. Note also there is a set of forces that act on body 2 and so forth.

The rest of basic dynamics is contained in what are generally called Newton's three laws of motion. The first law states that if a body has no net forces acting on it, it will continue in its present state of motion. This means that the velocity of an unforced body is unchanged; there is no acceleration. Newton took this idea from Galileo, see Section A.5.2.

In order to present the second law, we need the concept of mass. For our present purposes, we can take the simple definition of mass: it represents the amount of matter in an object. It was a difficult concept for Newton and the modern interpretations are also subtle. In its simplest form, Newton's second law states that a body responds to the presence of an unbalanced force by accelerating. The acceleration is the net force divided by the mass of the body, the famous $\vec{F} = m\vec{a}$. It is important to note that acceleration is a kinematic quantity and is defined once we have a length and a time.

Newton's third law states that if two bodies exert forces on one another, these forces are equal and opposite. The force of body two on body one is equal to the negative of the force of body one on body two, $\vec{F}_{21} = -\vec{F}_{12}$.

This law is also known as the law of action reaction. When this concept of force is a part of the interactions of bodies, this law is always true. With the development of Maxwell's theory of electromagnetism in which the field becomes a carrier of energy and momentum independent of sources, the idea of action reaction has to be modified, see Section 4.

It is very important to realize that, in Newtonian dynamics, if you know the forces acting on a body, either as a function of position or time, and you know the initial position and velocity, then you know the subsequent motion, i. e. the position as a function of time. This is the essence of causality. Given the initial position and velocity, and knowing the forces between all the bodies determines all the subsequent behavior of the bodies. It is important to note that this is a statement of the rules of dynamics that is local in space and time. Our new formulation of dynamics, action, will be a statement based on an assessment of the global properties of trajectories. The reconciliation of these two diametrically opposed approaches will be detailed later, Section **??**.

Although the Newtonian formulation of dynamics was a tremendous success, it actually had very limited applicability. As formulated, it applies only to point objects free to move in space. In fact, these laws as formulated by Newton can have only an approximate application to most of the cases that must be dealt with. This was fine when talking about the planets in orbit around the sun but, even for some of the simplest cases, these conditions do not hold. We will find that there is more to the world than just localizable point objects interacting with each other over a distance and that our description of dynamics and requirements for causality have to increase to account for all the phenomena observed in the universe, see Chapter 2.

Consider the problem of the motion of a blackboard eraser tossed into the air in the front of the lecture hall with a twisting spinning motion. Each part of the eraser is subjected to a huge array of forces. For convenience you can think of the parts of the eraser as the atoms but, even without an atomic hypothesis, all the following considerations still hold. Each part of the eraser is subject to the force of gravity and each part is subject to internal forces from the other parts of the eraser. First, there is an absurd number of parts and forces between the parts and between the parts and the world outside the eraser. In an eraser, since it is a macroscopic system, we could be dealing with the order of 10^{23} parts interacting with a like system, the earth, and thus have the order of 10^{23} equations. We simplify this situation somewhat by assuming that the effect of gravity is the same throughout the eraser and thus reduce these many gravitational forces to a single force acting at one point at the mass weighted center of the body. This is a good approximation for the case of a small eraser in the near vicinity of the earth. This still does not deal with the spinning and rotation of the eraser.

We know that, as the eraser rotates and spins, the different parts of the eraser will effect other parts. In fact, if the eraser was not a reasonably rigid body and held together by cohesive forces, in the spinning twisting motion, the parts would fly apart. Because the eraser is rigid, there are internal forces that act to hold the respective parts in a fixed relationship to each other. These forces are very complicated. They are in a very real sense unknowable; they are what they have to be to maintain the rigid configuration. These are called constraint forces. The eraser is not an exception. A car on the highway has a constraint force from the road called the normal force that is whatever it has to be to stop the car from falling into the road. Actually, with a little thought it becomes clear that almost all systems have constraints. In other words, the direct application of Newton's laws to systems that are constrained, which is most systems, is wrong or impossible. The fact that the motion of the eraser is given by a set of six dynamical equations, the center of mass motion and rotations, from a set of the order of 10^{23} is an impressive accomplishment.

In many special cases, special analysis was developed that allowed the use of statements like Newton's laws for motion in the presence of constraints and it was well known that the general problem of systems with constraints was a very important problem to both Newton and his immediate followers. The general problem of the motion of systems with algebraically described constraints was solved by Joseph-Louis Lagrange (1736 – 1813). This great contribution is based on a global measure over trajectories called the action. It is the modern basis for articulating the dynamics of any system and is the one that we will use. The advantage will be that the new rules will work in circumstances in which Newton's Laws were inappropriate or just did not make sense, Section **??**. Also with these new rules, we will find a more powerful understanding of the concepts of symmetry and be able to include all the modern constructs needed to clarify the world we live in. We will be able to use this new procedure to form a more solid understanding of the ideas of energy and momentum, Section **??**. In addition, action plays a foundational role in quantum mechanics. Because of these reasons and the role of action in expressing field dynamics, it is worthwhile linking these concepts.

A.2 Dynamics and Action

Dynamics, as stated above, are the rules for finding the temporal evolution of a system. In Newtonian Physics, this set of rules was succintly summed up in the rule: $\vec{F} = m\vec{a}$. In this section, we will find a new way to formulate the rules of dynamics that are more general but still produce the old $\vec{F} = m\vec{a}$ when it is appropriate. One complication will be that in order to formulate the rule, we will need ideas about kinetic and potential energy that are, in a sense, a throwback to the Newtonian approach. We will discover that the action is the fundamental concept and that ideas such as kinetic and potential energy are actually the derivative concepts. Before we are done though, these new ideas will take on a very different and more useful form. We will be able to understand why the massless photon has momentum but first we need to build the necessary background.

A.2.1 Background on Formulation of Action

The modern approach to dynamics, action, is based on the use of a global extremum principle. Action was not the first attempt to formulate rules on the basis of global extremum principles. In some sense, there seems to be a need in man to state the laws of nature as a maximum or minimum of some measure. We travel from place to place over least time, least traffic paths, or most picturesque routes. Similarly one of the earlier successful theories of optial phenomena was Fermat's (1601–1665) Least Time theory that light travels between two points in space over the path that takes the least time. In fact, Lagrange had predecessors in the formulation of the Principle of Least Action. Maupertuis (1698–1759) stated a least action principle as the basis for dynamics. His formulation though was not consistent in that it was reformulated for different circumstances. Euler (1707–1793) built on Mauperuis' work and used an exact statement of variation of trajectories to justify Newtonian dynamics. Lagrange was the first to realize the applicability of these methods as a technique to solve the general constraint problem, see [Yourgrau & Mandelstam 1968]

We describe the motion of anything as a connected set of events in space-time, a path in space-time called the trajectory of the particle. The events labeled by a place and a time and are the fundamental entities and a trajectory is a catalogue of the places where the object went as time evolves. an ordered set of contiguous events. We postulate that there is a globally defined physical quantity called the action and the naturally occurring trajectory in space-time is the one which has the least action. In some sense, this is an unfortunate name for this because we have used the word in another context, see Section 2.1, and it has a connotation in the conventional usage. Regardless, of the infinity of trajectories that can connect two events, the naturally occurring trajectory will turn out to be the one that has the least action. Obviously, before I can make this idea clear, we will need to back up a little as to why and to establish the terminology.

Consider a piece of chalk tossed up from my hand and returning to my hand some short time later. I am dealing with only one spatial dimension, up. The motion of the chalk is a continuous series of events that start with the toss and returns to my hand at a later time T. In between, the chalk has occupied a set of places at specific times in the interval, T. If you know the places for all times in that interval you have a trajectory. In Figure A.2, we show the trajectory in a space-time diagram.

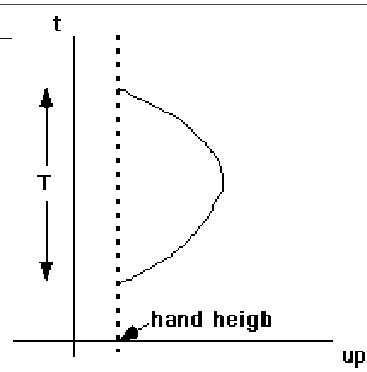

Figure A.2: **Trajectory of a tossed piece of chalk** Chalk tossed from my hand a caught in my hand at the same height at a time T later. In the naturally occurring trajectory, the chalk rises with decreasing velocity until it reaches a peak and then returns to the hand after the interval T.

For now, we deal with only trajectories that have the same start place and return place and the given time interval. A more general case would be release at one height and return to another height. Note that this kind of formulation of the situation of the motion does not follow the usual Newtonian formulation of the naturally occurring trajectory which specifies the trajectory develops from the start point and the velocity at the start and a force which in this case is given for all places and thus times. Because there is a force, the attraction of the earth for the chalk, there is an acceleration. Since there is an acceleration, the velocity changes. The velocity changes until it is reversed at the maximum height and starts to fall. While all this is happening, the chalk is tracing out a smooth arc in space time. In this Newtonian case, we specify the trajectory from the initial place and initial velocity and allow local in space time evolution accumulate to determine the trajectory. This description is very different than the one that we will be using for action. The action approach deals with the action over the entire trajectory and the beginning position and the start time and ending position and the finish time. This is a global approach to dynamics. On the surface it would seem to be difficult to reconcile these disparate approaches but you have to recover the Newtonian approach for the case in which the chalk can be treated as a point particle and free to move up and down without constraint.

The first issue of importance to us is the realization that the implementation of the rule requires that we know the action for all trajectories so that we can decide which one is the one with the minimum action. This implies that we must be able to calculate the action for all trajectories. Several possible trajectories are shown in Figure A.3. The next section will deal

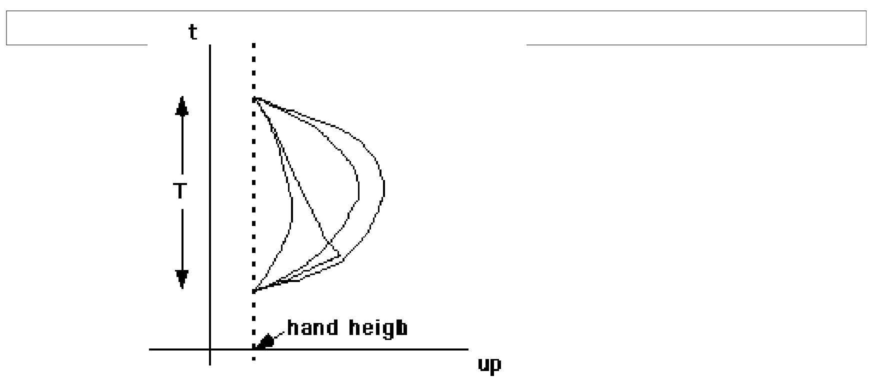

Figure A.3: **Possible trajectories for a tossed piece of chalk** There are an infinity of trajectories that can connect the event at the start of the toss with the event at the return of the chalk to the hand at a later time T.

with the complications that are implied by this requirement.

A.2.2 Mathematical Digression on the Number of Trajectories

In our articulation of Lagrange's Principle of Least Action in Section A.2.1, we casually assumed that it made sense to use the phrase "all possible trajectories" between two events. In a normal space-time, that's a lot of trajectories. To start with does it even make sense to identify "all trajectories". If you think about it, it means that somehow you produce an ordering for trajectories so that you can go through the lists to examine all possible cases. An ordering is a mapping of the paths onto an ordered set. Without much thought, it should be clear that there are a lot of trajectories – an infinity. Are there too many paths to order them like the integers? Two common examples of large sets are the integers for a discrete but infinite set or the points on a line for an infinite but continuous set.

Determining the size of infinite sets is a subtle issue. For instance, there are as many integers as there are odd numbers. That's because they can be ordered together – put into a one to one correspondence. This issue of the ability to order the trajectories is important to our ability to implement the Principle of Least Action.

How do you determine the number of trajectories? As stated above, you have to count them or compare their number with that of a set that is better understood. It turns out that there are a number of infinite sets, in fact, a countable infinity, The smallest of the standard sets of choice is the discrete infinite set that is the number of integers. Sets that have the same number of elements as this are relatively nice to deal with and once an identification with the numbers is established the elements can be manipulated like numbers. Sets of this size are said to be in the class $\aleph_0$. Anytime that you make a table, you are making a mapping between the set of integers and your set of objects that enter the table. If you have an ever larger set of objects, you have a set the size of $\aleph_0$ and you have ordered it with the integers.

In order to use the tools of analysis you need to deal with a system that has the right number of members. Functions are mappings of the real line onto the real line. The real line is, in fact, on example of the next larger infinite set, $\aleph_1$. It is bigger than the number of integers which also happens to be the same size as the number of rationales. The set made of all the points on the real line is the same size as the number of irrationales. By a simple ordering argument, we can show that the number of points on a line and the the number of points in a plane are the same. Again, an example of a property of these infinite sets that is not intuitive. In other words, there are as many points on one line as on any countable number of lines.

It is relatively straight forward to convince yourself that the number of trajectories is larger than the number of points on a line. This makes for a problem. Most of what we can do in analysis is dealt with through functions. By definition, functions are mappings of the real line on to the real line. Thus, our manipulations with trajectories cannot be considered functions and all the things that we learned about the manipulation of functions does not hold. In order to differentiate, mappings such as those on trajectory space onto the real line are called functionals.

Although the previous discussion is driven by our desire to understand the analysis required for the Principle of Least Action, all these statements about the transfinite number of trajectories holds for as mundane an idea as the number of paths connecting two points in space and, for example, issues of least time of travel. All these involve mappings that are functionals.

There are several ways to deal with this problem but the three simplest are the most common and which one is used depends on the use to which the action will be placed. The earliest and most commonly used was developed by Euler (1707–1783) even before Lagrange fully articulated the Principle of Least Action. He used a special analytic technique which used the idea of nearness in trajectory space and that a minimum should be stationary for nearby trajectories. His analysis is the basis for the calculus of variations. Modern descriptions of this approach thus use a "functional" derivative. This approach is covered in Section A.2.8. This approach is generally more mathematically sophisticated and we will defer its use until later.

The other approach uses techniques which take advantage of the global properties of the trajectory throughout and find methods to reduce the space of trajectories to a manageable set and then uses more conventional analysis. This approach is useful in cases where the action itself is important such in an quantum mechanics as formulated by the Feynman Path Integral. The most common use in this context is time slicing the trajectory, see Figure A.4. In this case, the time interval is segmented to reduce the definition of the trajectory to a set of positions, $\{x_i, t_i\}$, at each time slice t_i. This reduces the labeling of the trajectory to something that can be expressed on an $(\mathbb{R}^1)^n$ where n is the number of time slices. This reduces the analysis to something that can be handled with conventional mathematics.

A third approach is reduce the space of trajectories by using some reduced set. For instance if you know that the natural trajectory has a certain form, you can use trajectories that have that form. An example is given in Section A.2.5.

A.2.3 Definition of Action

Instead of $\vec{F} = m\vec{a}$ acting at each event on the trajectory to determine the motion of a body, there is now a new rule: minimize the action over the trajectory. In other words, nature chooses the least action trajectory from all the trajectories that share the same initial and final event. This is a formulation of motion that is very much like that of Fermat's Least Time formulation for the paths of light. To determine the trajectory, you pick two events, an initial event, x_0 and t_0, and a final event, x_f and t_f. There is a quantity called the action that is computed for every segment of every trajectory. The naturally occurring trajectory is the one that has the least action. This definition will be modified slightly later when realize the importance of symmetry and its natural language using action concepts, Section A.3.

The total action is defined as a sum of the actions in each segment of the trajectory. One important difference from formulation of Least Time for optical paths or Least Time of travel is that instead of taking segments along the path, for the calculation of the action, the trajectory is segmented in time slices, see Figure A.4. This gives a special role to the time variable. When you think about it you will realize that this is actually the way that time enters physics. The action in each time slice is found by weighting the time slice with a function of the positions and velocities called the Lagrangian. In this approach to dynamics, instead of trying to figure out what forces are causing the motion, you try to find what is the correct Lagrangian. In a real sense, when a modern physicist develops a new fundamental theory of some phenomena, it is by finding the correct Lagrangian so that the trajectory that yields the least action using that Lagrangian is the one that occurs naturally.

Also although we say all possible trajectories we will actually have to restrict the class of trajectories. One obvious restriction is to deal only with trajectories that advance in each time segment positively. In the case, of relativistic systems, and with great care, we will be able to lift this condition but that is a story for a different time. As in the case of least time in optics or travel, the size of the time slices depends on the trajectory and the precision required.

Once the trajectory and the Lagrangian are given the physics of the problem is set. For a simple object like the piece of chalk free to move up and down, the Lagrangian depends only on the position and velocity of the object and contains parameters such as the mass and the strength of the gravitational field. For more complex objects, there may be more dimensions allowed in the motion and more parameters that describe the object's world. Given the Lagrangian, the action is

$$S(x_f, t_f, x_0, t_0; traj.) \quad = \quad \sum_{traj.segments., x_0, t_0}^{x_f, t_f} L(x(t), v(t)) \Delta t$$

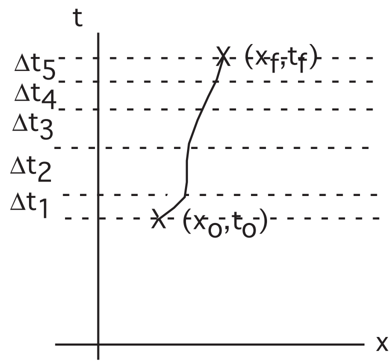

Figure A.4: **Trajectory for the computation of the action** In order to compute the action for a given trajectory, the trajectory is divided into time slice pieces. In this case, the trajectory is specified by the value of the position at each time slice. The trajectory is thus the set of $\{x_i\}$ at the time slice. Thus the space of trajectories has been reduced to $(\mathbb{R}^1)^n$ where n is the number of time slices. This requires only conventional mathematics for its analysis. For each segment of the trajectory, the positions and the velocity can be determined. The action is then computed for that time slice and the contributions of each time slice are added to produce the overall action. The sizes of the time slices are determined by the rate of change along the trajectory.

$$= \int_{t_0,traj.}^{t_f} L(x(t), v(t))\, dt \tag{A.1}$$

The rule that Lagrange and the procedures of least action require to reproduce $\vec{F} = m\vec{a}$ for unconstrained systems and also work for more general situations is that the Lagrangian, $L(x(t), v(t))$, should be the difference in the kinetic energy and the potential energy.

$$L(x(t), v(t)) = \frac{mv^2}{2} - V(x) \tag{A.2}$$

where $V(x)$ is the potential energy. For a proof see Section A.2.6.

Later, Section A.2.6, we will show how this reproduces Newton's laws. It is important to point out that, in this simple case of classical free particle motion, although this approach requires that you know the kinetic energy and potential energy, these concepts are actually derived from the actions and not the other way; for modern physics, it is the action which is the fundamental construct and concepts such as energy and momentum are derivative. From the above, it seems that you need to know the potential energy before you can write the Lagrangian. This is only for historical and pedagogical reasons. When a modern physicist is struggling with understanding some basic new phenomena, it is the other way around. We start with a Lagrangian and then see what the consequences are. The idea that action is the basis of all that we can know will also turn out to be the basis for the idea of a unified theory. Since the action is the basis of all dynamics, the idea that disparate phenomena can be joined in a single unified theory is that there is a single action statement for all the phenomena. It is not enough to merely write the equations all together. In modern language, Maxwell unified the electric and magnetic forces because the entire ensemble of equations is derivable from a single Lagrangian and the least action principle.

A.2.4 Some Simple Examples

Trajectory of a Free Particle

To test our new dynamic, let's look at the simplest situation possible – a free particle. A free particle is one that is free to move without any hinderance; it has no forces acting on it. All places have the same energy value and thus $V(x) = 0$. We know from Newton's first law, Section A.1, that, in this case the particle travels in a straight line at a constant speed.

Using Lagrange's rules to get the solution for the free particle, we chose the Lagrangian that is just the kinetic energy or $L(v(t)) = \frac{mv^2}{2}$. To make it even simpler, let's require that the released particle is to return to the original position after a time T. This is actually the general case but we will discuss that later, Section A.5.2 The action is

$$S(0,0,0,T,traj.) = \sum_{traj.,0,0}^{0,T} \frac{mv^2}{2}\Delta t \qquad \text{(A.3)}$$

Note that the action is a positive definite quantity for all velocities. Therefore any trajectory that has a non-zero velocity anywhere in the time interval will have a positive action. The trajectory that has $v(t) = 0$ for all t in the interval has an action of zero. This is clearly a minimum of the action since all other trajectories will have a positive action. Thus, this is the natural path. Actually any Lagrangian with v^2 in it will accomplish the same thing. The m is in it to give it the correct dimensions and the 2 for historical reasons. In fact, the m that is in the Lagrangian is the definition of mass. More on this later, see SectionSec:Mass. shinola

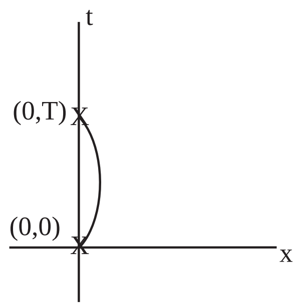

Figure A.5: **Space-time diagrams for the action for a free particle** A particle with no forces acting on it moves between two events,$(0, 0)$ and $(0, \mathrm{T})$. A possible trajectory is shown. Our experience with force free motion is that the straight line trajectory is the one that nature chooses; the particle remains at the point of release.

Using this same result and remembering the material on Galilean invariance in Section **??**, we can solve a more general problem. Suppose we have a free particle that moves through the two events $(0, 0)$ and (x_f, t_f). Again, since the particle is free, the natural trajectory is the straight line connecting these events. To an observer moving by us at a speed of $v = \frac{x_f}{t_f}$, the object is a rest during the entire time interval. To that observer it is free and the initial an final events are $(0, 0)$ and $(0, t_f)$ and the natural path is the straight line along the t axis as before. Thus to us the natural trajectory will be the straight line with slope $fract_f x_f$. Let's obtain this same result with a direct analysis.

Consider a general trajectory connecting events $(0, 0)$ and (x_f, t_f), see Figure A.6. Our problem is to find all possible trajectories between these events and then, for each trajectory, find the action. As we discussed about paths when dealing with the Fermat's least time approach to optics in Section A.2.2. path space is a rich mathematical structure. We want to do analysis. To do analysis we have to reduce the complexity of path space to something that can be described by functions. There are all these same difficulties when dealing with trajectories. To simplify our trajectory space, we reduce the trajectories that we consider to those that are "once kinked". Place the kink along the line $t = \frac{t_f}{2}$, see Figure A.6. In this reduced space, trajectories can be labeled by the distance, a, of the kink from the event $(\frac{x_f}{2}, \frac{t_f}{2}$ along that line. Using this trajectory in the appropriately modified Equation A.3 to take account of the new ending event, and the fact that the inverse slope of the line is the velocity in that segment, it is easy to compute

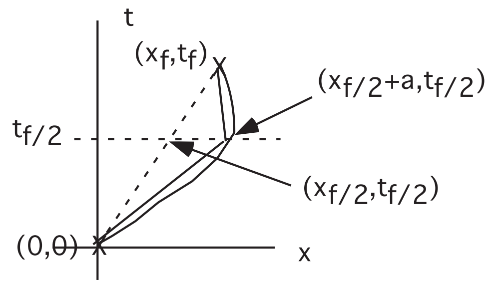

Figure A.6: **Space-time diagrams for the action for a free particle that changes position** A particle with no forces acting on it moves between two events,(0, 0) and (x_f, t_f). A possible trajectory is shown. The general trajectory connecting these events would be very difficult to describe. We will approximate the trajectory with a trajectory that is kinked at the mid-time and straight otherwise.

the action for the trajectory labeled a. It is

$$S(0,0,x_f,t_f,traj=a) = \frac{m}{2}\left(\frac{\left(\frac{x_f}{2}+a\right)^2}{\frac{t_f}{2}} + \frac{\left(\frac{x_f}{2}-a\right)^2}{\frac{t_f}{2}}\right). \qquad \text{(A.4)}$$

This is an even function of a and thus has a minimum at $a = 0$. This confirms our result that the natural trajectory, the constant velocity trajectory, is the least action trajectory.

A.2.5 Examples of action

Gravitation above a Flat Earth

As a simple example that we are all familiar with, consider the case of motion above the surface of the earth. Here the energy of position, the potential energy, is due to the gravitational interaction of a massive body with the earth. For this case, the potential energy at a height h above the earth is $V(\vec{r}) = -\frac{Gm_e m}{R_e+h}$, where m_e is the mass of the earth, m the mass of the body, and R_e is the radius of the earth. For motion near the surface, a few meters up or down, from "Things Everyone Should Know," Section **??**, we can use $(1+x)^n \approx 1 + n\ x$ for $x \ll 1$ to reduce this to

$$V(h) = -m\left(\frac{Gm_e}{R_e}\left(1-\frac{h}{R_e}\right)\right) = V(R_e) + mgh,$$

where we recognize $g = \frac{Gm_e}{R_e^2}$. Since this potential is to be used in an action, as we will see later in Section A.5, changing the action by a constant does not change the physical results in a significant way, we can drop the $V(R_e)$ term. This reduces the potential energy for objects moving in the near vicinity of the earth to

$$V(h) = mgh. \tag{A.5}$$

Another way to look at this result is to say that for motion restricted to be near the surface of the earth, the earth appears as an infinite plane. In this case, the force of gravity above the plane can not depend on anything, in particular, the height above the plane or the position sideways over the plane. Thus the force also can only be toward or away from the plane. Then realizing from the analysis above in Section A.2.6 that the change in potential as you change position is the force, the only form for the potential in this case is $mgh +$ constant.

For now let us consider only up and down motion, not any sideways motion. The potential energy is mgh where h is the height. Thus the action for any trajectory between an initial height, h_0 at time t_0 and final height, h_f at time t_f is

$$S(h_0, t_0, h_f, t_f; traj.) = \sum_{traj., h_0, t_0}^{h_f, t_f} \left(\frac{mv^2}{2} - mgh \right) \Delta t \tag{A.6}$$

where the path is given by $h(t)$. Note that if you know $h(t)$, you also know $v(t)$. You can see from the form of the action that you will lower the action by having $h(t)$ to be at large h for as much time as possible. The problem is that since the initial and final position and time are given, it takes high velocity to get to large h. The high velocity increases the action. $\Longrightarrow$ There is a single least action path. This is the trajectory that the particle follows.

Let's get more specific. This is again the problem of a piece of chalk tossed up in the air. First the simplest case, the chalk is released and returns to the same height after a t time T.

We need to study the action for all trajectories connecting these events. Again, because of the complexity of the idea of all trajectories, we will need to reduce the number of trajectories. A first step is to use our experience to limit ourselves to simple trajectories that rise smoothly to a peak at some height a at which time the velocity is zero and then returns over a trajectory that is a reflection of the one on the rise. Our natural trajectory must be in that family. This is still a very rich family and too rich to do analysis. This is the same problem that we had with the Fermat's Least Time, Section A.2.2,

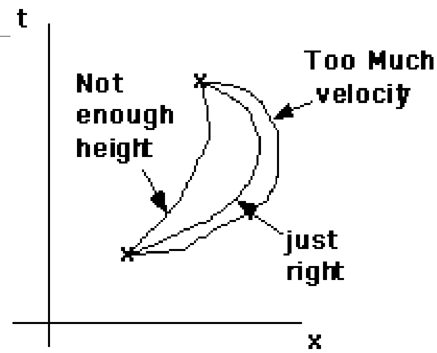

Figure A.7: **Trajectory for Particle in Uniform Gravitational Field** Space-time diagrams for calculation of the action for a particle in a uniform gravitational field. The least action trajectory is just the right compromise between too much kinetic energy and some potential energy.

and the free particle, Section A.2.4. As in the latter case, the once kinked path can be used to approximate the family of smooth trajectories that have these properties, see Figure A.8. Here again the variable a is the height of the approximate trajectory but more importantly now it is a label that can be used to specify the particular trajectory from the family with which we are dealing.

Since this approximate trajectory is broken line segments, it is relatively easy to compute the action.

$$S(0,0,0,T;traj.) = \sum_{(0,0)\ \text{traj.}}^{(0,T)} \left(\frac{mv^2}{2} - mgh\right)\Delta t. \tag{A.7}$$

For a straight line path, v is a constant and is the inverse slope of the line, and is $\frac{a}{\frac{T}{2}}$ in magnitude for both segments. The height is a more subtle question since it varies with time from 0 to a. Being reasonable, we can use the average height, $\frac{a}{2}$. For the sophisticates among you, there is the problem that the concept of average is a not trivial, see Section **??**. Thus the action for the first segment is

$$S_1(T,a) = \frac{ma^2}{2\left(\frac{T}{2}\right)^2}\frac{T}{2} - \frac{mga}{2}\frac{T}{2}. \tag{A.8}$$

Note that once I have made a mapping of the paths onto the line that S becomes a regular function of the path label, a, instead of a functional.

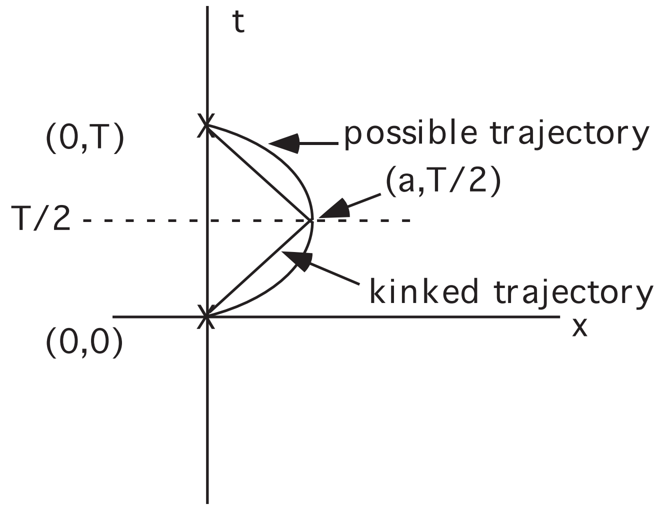

Figure A.8: **Possible trajectory for the action for a particle in a uniform gravitational field** A piece of chalk is tossed upward and caught later at the the same height. A possible trajectory is shown. The natural trajectory is one from the family of smooth trajectories that rise to a peak at a height a smoothly and then return to a lower height on a reflected trajectory. This is still a large family of trajectories. We can approximate the members of this family with a once kinked trajectory with the same height at the time $\frac{T}{2}$.

Although the velocity is negative, since only v^2 enters the lagrangian, the action on the second segment is the same and the total action is

$$S(T,a) = 2S_1(T,a) = ma\left(2\ \frac{a}{T} - \frac{g}{2}T\right) \tag{A.9}$$

This has zero's at $a = 0$ and $a = \frac{gT^2}{4}$. The dependence of the action on the path label a is shown in Figure A.9. I have used dimensions in which $g = T = 1$.

We can see that there is a minimum half way between the two zero's at $a = 0$ and $a = \frac{gT^2}{4}$. This implies that the trajectory from this set that is the least action trajectory is the one with

$$a_{least\ action} = \frac{gT^2}{8}. \tag{A.10}$$

Since this is not only the path selecting parameter but is also the height, we get that the height is $\frac{gT^2}{8}$.

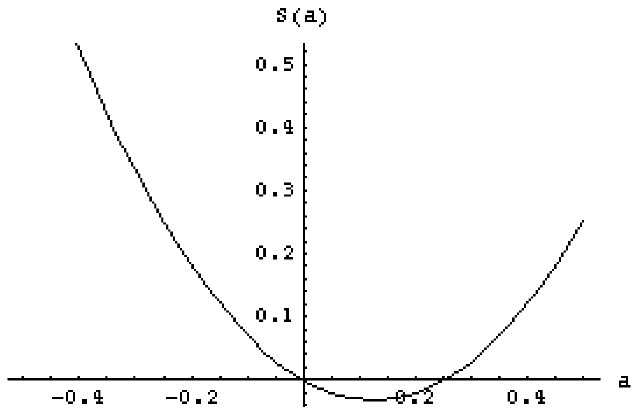

Figure A.9: **Action as a function of** a The action as a function of of the trajectory label a. This curve is a combination of a parabola, $\frac{2m}{T}a^2$, concave up with its vertex at the origin and a straight line, $-\frac{mgT}{2}a$, with negative slope through the origin.

Same Example done another way

I am going to do some mathematics here that I do not expect that you will be able to reproduce. I do this to show you that it can be done and that the ideas of mathematics are useful. You are not expected to do integrals and take derivatives although you should be able to follow a development using them.

Once again, we want to examine the case of an object of mass m moving in the vicinity of the earth. We can also guess that the correct answer for the height as a function of time is a parabola, all parabolas that fit the time interval are of the form $h(t) = at(t-T) \Rightarrow v(t) = 2at - aT$, where a is label of the path in path space. In this case, a has the dimension of an acceleration, $L \overset{\text{dim}}{=} a \times T^2$ or $a \overset{\text{dim}}{=} \frac{L}{T^2}$.

The Lagrangian is $L = \frac{1}{2}mv^2 - mgh$ and the action is

$$\begin{aligned} S &= \int_{(x_0,t_0),\text{Path}}^{(x_f,t_f)} \Big(\frac{1}{2}mv^2 - mgh\Big)dt \\ &= m\int_0^T \Big(\frac{1}{2}(2at-aT)^2 \ - gat(t-T)\Big)\, dt \\ &= m\Big(\frac{a^2T^3}{6} + \frac{1}{6}agT^3\Big) \end{aligned}$$

This can be factored to $S = \frac{mT^3}{6}a(a+g)$.

To find the minimum, we can again realize that there are two zeros of S(a). One at $a = 0$ and one at $a = -g$. The minimum is half way between them at $a_{least\ action} = -\frac{g}{2}$

Otherwise, we can take the derivative of S(a) with respect to a and set it equal to zero. Thus

$$\begin{aligned} \frac{dS}{da} &= \frac{d}{da}\left(\frac{mT^3}{6}a(a+g)\right) \\ &= \frac{1}{6}amT^3 + \frac{1}{6}(a+g)mT^3 \\ &= \frac{1}{6}\ (2\ a+g)\ m\ T^3 \end{aligned} \tag{A.11}$$

or requiring

$$\frac{dS}{da} = 0$$

implies that $a_{least\ action} = -\frac{g}{2}$ is the natural trajectory. In Figure A.10, note how the action varies with a. Again I have used units with $g = T = 1$.

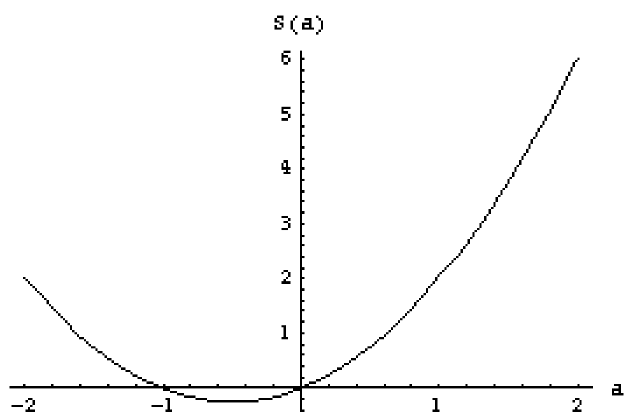

Figure A.10: **Action as a function of** a **as an acceleration** Action as a function of a when the parameter a has the dimensions of an acceleration. This example shows that the trajectory label does not have to be a height.

It is very important to realize that we have now shifted our definition of the natural trajectory. We have been using the definition that the natural trajectory is the one that minimizes the action. Now we are using a definition which says that the naturally occurring trajectory is the one for which the action is slowly varying,

$$\delta S = 0, \tag{A.12}$$

for trajectories in the neighborhood of the naturally occurring trajectory. For this particular example, this is equivalent in to our earlier definition. This is because our action is smoothly varying in trajectory space and we want a soft minimum. It will turn out that the condition Equation A.12 is the one that is actually used in all cases in classical mechanics and that quantum mechanics actually provides a basis for the idea that Equation A.12 is the more fundamental statement, see Section **??**.

A.2.6 Least Action Reproduces Newtonian Physics

In Section A.2.3, we indicated that you could recover the usual statement of mechanics known as Newton's laws by defining the Lagrangian as the kinetic energy minus the potential energy. In this section, we will prove this. We will use the most simple and direct method for controlling the problem of the size of trajectory space by using a simple time sliced reduction of the trajectories to an $(\mathbb{R}^1 \times \mathbb{R}^1)^n$ by labeling the trajectory with the set $\{x_i, t_i\}$, where x_i is the position on the trajectory at the time slice t_i and n is the number of time slices and is a large number.

We are interested in all possible trajectories that connect an initial event (x_0, t_0) and a final event (x_f, t_f), see Figure A.4. Consider the case of n time slices. Any trajectory that connects the initial and final events is labeled by the set of numbers $\{x_i, t_i\}$. We construct the action in the prescribed fashion, $L = m\frac{v^2}{2} - V(x)$. As in the earlier examples, the velocity in any segment, which is approximated by the straight line segment between two time slices, is the change in position in the segment divided by the time interval, $\frac{x_i - x_{i-1}}{t_i - t_{i-1}}$. Similarly, the position in the segment is the average position over that segment[1], $\frac{x_i + x_{i-1}}{2}$. Thus following the first of Equations A.1, the action for any trajectory is

$$\begin{aligned} S(x_f, t_f, x_0, t_0; \{x_i, t_i\}) &= \sum_{seg.\ \{x_i, t_i\}, x_0, t_0}^{x_f, t_f} \left(\frac{m}{2} \left(\frac{x_i - x_{i-1}}{t_i - t_{i-1}} \right)^2 \right. \\ &\qquad \left. -V\left(\frac{x_i + x_{i-1}}{2} \right) \right) (t_i - t_{i-1}) \,, \end{aligned} \tag{A.13}$$

[1] Depending in the nature of the potential function a differently weighted average may be more appropriate but if the time slices are small enough this simple average is adequate.

where, as indicated, we sum over all the segments between (x_0, t_0) and (x_f, t_f).

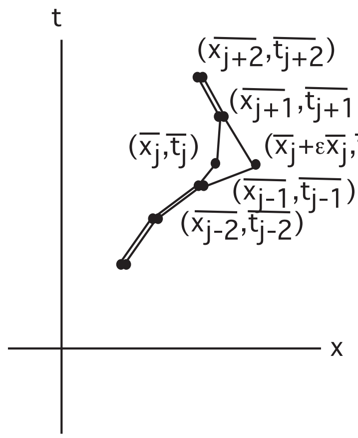

Figure A.11: **Natural Trajectory and Nearby Trajectory** In the time sliced formulation of action, trajectories are labeled by the set $\{x_i, t_i\}$ where i ranges from 0 to n where n is the number of time slices. The naturally occurring trajectory is the one that minimizes the action and is designated by the $\{\bar{x_i}, \bar{t_i}\}$. The condition of minimum is established by requiring that the change in the action between the naturally occurring trajectory and any nearby trajectory vanish to first order in the measure of nearness, see Equation A.12. A measure of nearness for trajectories is to increment the space component of a trajectory at some time slice by a small amount. Above the natural trajectory which has many time slices is incremented at the j th slice to produce the nearby trajectory. In the calculation of the difference of the action over the naturally occurring and nearby trajectories only the parts of the trajectories around the j th slice are relevant. This converts a global condition of naturally occurring to one that is local in time like the Newtonian statement of mechanics.

Assuming that, from the very large set $\{x_i, t_i\}$, there exists one that minimizes the action and is thus the naturally occurring trajectory. Designating this trajectory $\{\bar{x_i}, \bar{t_i}\}$, we can calculate the action with Equation A.13. We can also examine the nearby trajectory which is the same as $\{\bar{x_i}, \bar{t_i}\}$ everywhere except at some particular time slice for the natural trajectory, $\bar{t_j}$, and, for this new trajectory, the position coordinate differs from the naturally occurring one by a small amount. In other words, the new nearby trajectory is the one for which $\{x_i, t_i\} = \{\bar{x_i}, \bar{t_i}\}$ except when $i = j$ and, for that case, the position time pair is $(x_j, t_j) = (\bar{x_j}(1+\epsilon), \bar{t_j})$ where ϵ is a small number, $\epsilon \lll 1$. ϵ is such a small number that we can neglect all powers of ϵ greater than one. As we will see the ϵ^0 terms will drop out of the change in the action and thus we need only deal with the ϵ^1 term.

From Figure A.11, it is clear that the difference in the action of the naturally occurring trajectory and the nearby one requires only calculating the segment of the trajectories before and after the incremented one. This not only simplifies the calculation but also shows how this use of near in trajectory space can be used to convert a global statement of dynamics to one that is local in time like Newton's laws, see Section A.2.1.

Writing the difference in the actions for the naturally occurring trajectory and the nearby one,

$$
\begin{aligned}
\delta S &= S_{\text{near by}} - S_{\text{naturally occurring}} \\
&= \left\{ \frac{m}{2} \left(\frac{\bar{x}_j(1+\epsilon) - \bar{x}_{j-1}}{\bar{t_j} - \bar{t}_{j-1}} \right)^2 - V\left(\frac{\bar{x}_j(1+\epsilon) + \bar{x}_{j-1}}{2} \right) \right\} (\bar{t_j} - \bar{t}_{j-1}) \\
&\quad + \left\{ \frac{m}{2} \left(\frac{\bar{x}_{j+1} - \bar{x}_j(1+\epsilon)}{\bar{t}_{j+1} - \bar{t}_j} \right)^2 - V\left(\frac{\bar{x}_{j+1} + \bar{x}_j(1+\epsilon)}{2} \right) \right\} (\bar{t}_{j+1} - \bar{t}_j) \\
&\quad - \left\{ \frac{m}{2} \left(\frac{\bar{x_j} - \bar{x}_{j-1}}{\bar{t_j} - \bar{t}_{j-1}} \right)^2 - V\left(\frac{\bar{x_j} + \bar{x}_{j-1}}{2} \right) \right\} (\bar{t_j} - \bar{t}_{j-1}) \\
&\quad - \left\{ \frac{m}{2} \left(\frac{\bar{x}_{j+1} - \bar{x}_j}{\bar{t}_{j+1} - \bar{t}_j} \right)^2 - V\left(\frac{\bar{x}_{j+1} + \bar{x}_j}{2} \right) \right\} (\bar{t}_{j+1} - \bar{t}_j) \,. \qquad \text{(A.14)}
\end{aligned}
$$

In order to proceed, we will use Newton's approximation, $f(x+\epsilon) \approx f(x) + \epsilon \frac{df}{dx}(x)$ which is valid for any small ϵ and any smooth fx for the potential function in the near by trajectory terms. Expanding and dropping terms of order ϵ^2,

$$
\begin{aligned}
\delta S &= \epsilon x_j \left\{ m \left\{ \frac{\bar{x}_j - \bar{x}_{j-1}}{\bar{t}_j - \bar{t}_{j-1}} - \frac{\bar{x}_{j+1} - \bar{x}_j}{\bar{t}_{j+1} - \bar{t}_j} \right\} \right. \\
&\quad - \frac{1}{2} \left\{ \frac{dV}{dx} \left(\frac{\bar{x}_j + \bar{x}_{j-1}}{2} \right) (\bar{t}_j - \bar{t}_{j-1}) \right. \\
&\quad \left. \left. + \frac{dV}{dx} \left(\frac{\bar{x}_{j+1} + \bar{x}_j}{2} \right) (\bar{t}_{j+1} - \bar{t}_j) \right\} \right\} \qquad \text{(A.15)}
\end{aligned}
$$

This result can best be interpreted by first multiplying through by $\frac{\bar{t}_{j+1} - \bar{t}_{j-1}}{\bar{t}_{j+1} - \bar{t}_{j-1}}$, and reorganizing

$$\begin{aligned}\delta S &= \epsilon x_j\left(\bar{t}_{j+1}-\bar{t}_{j-1}\right)\left[-m\frac{1}{\bar{t}_{j+1}-\bar{t}_{j-1}}\left\{\frac{\bar{x}_{j+1}-\bar{x}_j}{\bar{t}_{j+1}-\bar{t}_j}-\frac{\bar{x}_j-\bar{x}_{j-1}}{\bar{t}_j-\bar{t}_{j-1}}\right\}\right. \\ &\quad -\frac{1}{2}\left\{\frac{dV}{dx}\left(\frac{\bar{x}_j+\bar{x}_{j-1}}{2}\right)\left(\frac{\bar{t}_j-\bar{t}_{j-1}}{\bar{t}_{j+1}-\bar{t}_{j-1}}\right)\right. \\ &\quad \left.\left.+\frac{dV}{dx}\left(\frac{\bar{x}_{j+1}+\bar{x}_j}{2}\right)\left(\frac{\bar{t}_{j+1}-\bar{t}_j}{\bar{t}_{j+1}-\bar{t}_{j-1}}\right)\right\}\right]. \qquad \text{(A.16)}\end{aligned}$$

For $\delta S = 0$, since all the factors in front of the square bracket are non-zero, the terms inside the square bracket must vanish at each t_i. This is a condition on the naturally occurring trajectory. The terms in the curly brackets on the first line of Equation A.16 are the change in velocity in the two relevant segments of the trajectory which when divided by the time interval over the two segments is the acceleration at t_j. The mass times this acceleration must equal the remaining terms in the curly bracket. These are position and time segment weighted averages of the quantity $\frac{dV}{dx}$ evaluated in the two relevant segments. Since in Newtonian mechanics $F = -\frac{dV}{dx}$, we have that for each i along the naturally occurring trajectory

$$m\bar{a}_i = -\frac{dV}{dx}(\bar{x}_i) = F.$$

A.2.7 More Examples of Actions

Scattering

Two particles, one of mass m_1 and the other of mass m_2 collide. After the collision, the particles move away from each other, both still with masses m_1 and m_2. This is a very special problem whose important cannot be over emphasized. In a very real sense, when we probe the nature of the elementary constituents of matter, scattering experiments are the primary source of our knowledge. In addition, the process is so basic that it will allow us to begin to better understand many fundamental issues.

How do we handle this process? First, we have to decide what is meant by two independent particles. Before the particles make contact, they move as if the other particle was not present, i. e. they are independent. It is reasonable therefore to assume that while they are apart or not interacting, the two particles actions add and are the usual free particle action. In other words, there is a free particle action the tells you all the properties of what is meant by a particle and its nature. For our construction of the action of the free particle in Section A.2.4, we used the Lagrangian $L(x, v) = \frac{mv^2}{2}$. The Lagrangian says the the object identified as a free particle does not treat different places differently and thus there is no x dependence in the Lagrangian. If we want to recover Newton's Law, see Section A.2.6, we use the usual classical kinetic energy. We will find that in other circumstances, for instance for a rapidly moving particle, Section **??**, that a different free particle Lagrangian is appropriate. If we wanted to describe something more complicated than a point particle, say a small rod, we would need elements that deal with what a rod is such as moment of inertia and directional variables.

By using as the action the sum of the single particle actions, the properties of the total system will be the sum of the properties of the parts. If we did this though, and this was the end of it, nothing interesting would ever happen; the particles would merely pass through each other unchanged in their motion. We want them to scatter. Thus in addition, we need to add a part that carries the interaction. The interaction will have a Lagrangian that is made up of relationship variables such as their separation in addition to the particle labels. In other words, the action is made up of the following parts:

$$\begin{aligned} \text{Total Action} &= \text{Free Action(variables particle 1)} \\ &+ \text{Free Action(variables particle 2)} \\ &+ \text{Interaction Action(variables particle 1,} \\ &\quad \text{variables particle 2, relationship variables).} \end{aligned} \tag{A.17}$$

Of course, it is actually redundant to list the relationship variables in the interaction action since they will be composed of the variables of particle 1 and 2 anyway. The importance of displaying the relationship variables separately is to be able to say that, for a scattering situation, the interaction action is zero when the relationship variables such as the separation are large. In a collision, we assume that most of the time the particles travel toward or away from each other and that the interaction terms contribute only for a short time when the particles are in contact and thus this interaction term is small and does not add significantly to the total action of the process. Another point to note is that, since the interaction terms are dominated by the relationship variables, the contribution from the interaction action should be independent of where and when the collision takes place. Thus, we can write the action for this simple one dimensional scattering process as

$$S = \sum_{(x_{10},t_{10}),Path}^{(x_{1f},t_{1f})} m_1 \frac{{v_1}^2}{2}\Delta t + \sum_{(x_{20},t_{20}),Path}^{(x_{2f},t_{2f})} m_2 \frac{{v_2}^2}{2}\Delta t + A, \tag{A.18}$$

where A represents the interaction action. The scattering process is shown in Figure A.12.

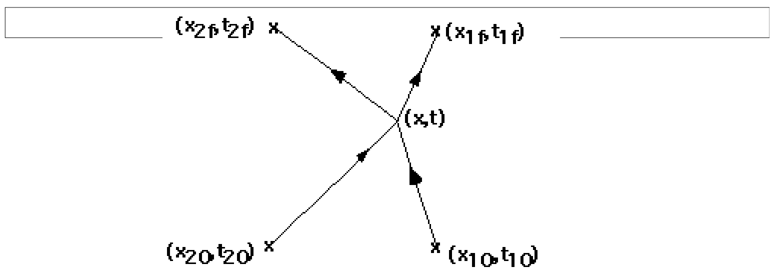

Figure A.12: **Space-time diagram for a scattering event** Two particles of mass m_1 and m_2 free to move in one spatial dimension are directed at each other and collide at the event (x, t) and then move apart . A space-time diagram for a scattering event with particle one starting at event (x_{1_0}, t_{1_0}) and returning to (x_{1_f}, t_{1_f}) and particle two starting at event (x_{2_0}, t_{2_0}) and returning to (x_{2_f}, t_{2_f}) is shown. Although all trajectories connecting the initial and final events and the collision event should be examined, we know that free particles have a natural trajectory that is a straight line, see Section A.2.4.

We want to do all paths but we know that the straight path is the least action for a free particle and so all we need to do is use straight paths between the initial and collision and collision and final events. We can immediately write down the action as a function of the position and time of the collision. The coordinates of that event are the only free parameters in the problem.

Note that we are being consistent in our use of action. When you talk about collisions in the general physics class you set the initial velocities. Here we use the initial and final events. Evaluating the free particle actions, for this system of trajectories, the action is

$$S = \frac{m_1}{2}\frac{(x - x_{1_0})^2}{(t - t_{1_0})} + \frac{m_2}{2}\frac{(x - x_{2_0})^2}{(t - t_{2_0})} + \frac{m_1}{2}\frac{(x_{1_f} - x)^2}{(t_{1_f} - t)} + \frac{m_2}{2}\frac{(x_{2_f} - x)^2}{(t_{2_f} - t)} + A. \tag{A.19}$$

We want to find the trajectory that has the least action and since we have now reduced the world of trajectories to the label of the collision point,

x and t. Thus we need to minimize this in what are now the labels, x and t. You could plot this and find the minimum by hand , see Figure A.13, but, if you allow me to use calculus, I can find a simple analytic expression for the $x = x_{min}$ and $t = t_{min}$ that yields the least action. This means taking the derivatives with respect to x and t and finding the value of x and t that satisfy $\frac{\partial S}{\partial x} = 0$ and $\frac{\partial S}{\partial t} = 0$. This x and t label the naturally occurring trajectory.

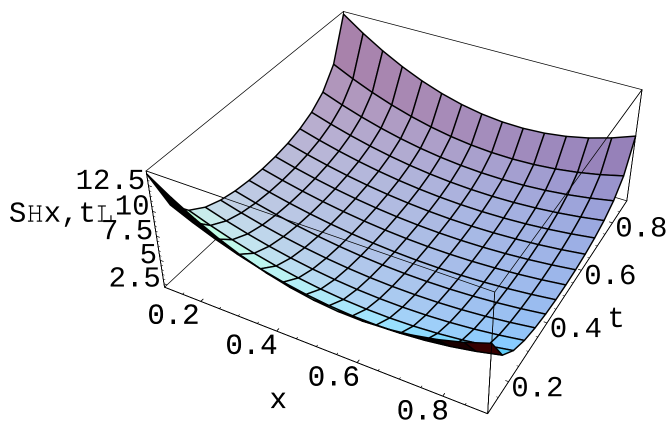

Figure A.13: **Action for a Scattering Event** Action as a function of x and t for a scattering event shown in Figure A.12. There is a clear minimum and it occurs at the points at which Equation A.21 and Equation A.23 are satisfied.

Take my word for it. The condition for a minimum in x is

$$m_1 \frac{(x_{min} - x_{1_0})}{(t_{min} - t_{1_0})} + m_2 \frac{(x_{min} - x_{2_0})}{(t_{min} - t_{2_0})} - m_1 \frac{(x_{1_f} - x_{min})}{(t_{1_f} - t_{min})} - m_2 \frac{(x_{2_f} - x_{min})}{(t_{2_f} - t_{min})} = 0 \tag{A.20}$$

or

$$m_1 \frac{(x_{min} - x_{1_0})}{(t_{min} - t_{1_0})} + m_2 \frac{(x_{min} - x_{2_0})}{(t_{min} - t_{2_0})} = m_1 \frac{(x_{1_f} - x_{min})}{(t_{1_f} - t_{min})} + m_2 \frac{(x_{2_f} - x_{min})}{(t_{2_f} - t_{min})} \tag{A.21}$$

Realizing that momentum is mv in classical physics and that v is the difference in positions divided by the the differences in times, this is the statement that the momentum into the collision is equal to the momentum out of the collision.

The condition that there is a minimum in t gives

$$\frac{m_1}{2}\frac{(x_{min}-x_{1_0})^2}{(t_{min}-t_{1_0})^2}+\frac{m_2}{2}\frac{(x_{min}-x_{2_0})^2}{(t_{min}-t_{2_0})^2}-\frac{m_1}{2}\frac{(x_{1_f}-x_{min})^2}{(t_{1_f}-t_{min})^2}-\frac{m_2}{2}\frac{(x_{2_f}-x_{min})^2}{(t_{2_f}-t_{min})^2}=0 \tag{A.22}$$

or

$$\frac{m_1}{2}\frac{(x_{min}-x_{1_0})^2}{(t_{min}-t_{1_0})^2}+\frac{m_2}{2}\frac{(x_{min}-x_{2_0})^2}{(t_{min}-t_{2_0})^2}=\frac{m_1}{2}\frac{(x_{1_f}-x_{min})^2}{(t_{1_f}-t_{min})^2}+\frac{m_2}{2}\frac{(x_{2_f}-x_{min})^2}{(t_{2_f}-t_{min})^2} \tag{A.23}$$

Which is the same as the statement that the energy into the collision event is equal to the energy out of it.

Figure A.13 shows the action as a function of the position and time of the collision event. This is for the case that $\frac{m_2}{m_1}$is 1.5 and the original and final events for particle 1 are (0,0) and (0,1) and for particle 2 are (1,0) and(1,1).

This exercise also gives us an interesting insight on what mass is. In an early assignment in this course, you were asked to devise a method for measuring mass that does not relay on gravity. Some of you came up with the idea of using collisions to define a mass scale. You can see that this analysis is directly relevant to that kind of definition. In the construction of the action, for the case of the single particle, mass is an overall factor; it is the thing you put in front of the v^2, in the action. If the world consisted of only one particle, mass would be irrelevant since all it does is multiply the action. The process of finding the natural trajectory is unchanged by the an overall scale factor on the action. Mass becomes interesting only when you have more than one particle. If there is more than one particle, you can not remove all the masses with a single scaling factor. The ratios of the mass remain. Consider a scattering event between two particles with the initial and final positions of the two particles the same before and after the collision. If the particles had equal masses, the position of the collision event is at the center. The trajectories of both particles are equally kinked. On the other hand, the higher the mass ratio of say the second particle, the less the trajectory associated with that particle will kink when it collides with another particle. In the limit of a very large mass second particle, there is no bending of the second trajectory and it looks like the first particle has hit a brick wall. This is the essence of inertia.

A.2.8 Euler-Lagrange Approach

Up until now, we have used minimal analytic tools to study the implications of the Principle of Least Action. Not only is it useful to take advantage of the full array of the tools of modern analysis, many of these tools were developed in order to articulate the implications of these principles. The first use of these techniques were developed by Euler and the basic idea is the basis of most studies of the Principle of Least Action. We will use a language that is a bit more careful about the analysis but is in the spirit of Euler's analysis. The idea is similar to the finite segment approach of Section A.2.6 but the language is more subtle.

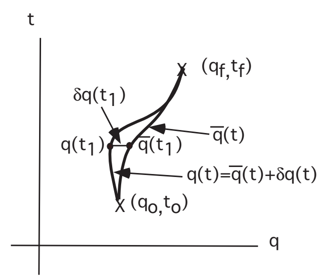

Figure A.14: **A Nearby Action Trajectory** Assuming that a natural trajectory, $\bar{q}(t)$, that connects two events, an initial event, (q_0, t_0), and a final event, (q_f, t_f) exists. That trajectory minimizes the action. The action for a "nearby" trajectory, $q(t) = \bar{q}(t) + \delta q(t)$, should then produce an action that differs from the natural trajectory action by a small amount. The difference between the action of this nearby trajectory and that of the action on the natural trajectory should be small and proportional to the differences between the natural and nearby trajectory, $\delta q(t)$. Since this difference in the actions must vanish for all possible small nearby trajectories the coefficient of the difference in the trajectories must be zero.

Consider the case of a single point particle, moving between the event (q_0, t_0) and the event (q_f, t_f), see Figure A.14. The action is

$$S[q(*)](q_0, t_0; q_f, t_f) \equiv \int_{t_0}^{t_f} L\left(\dot{q}(t), q(t); t\right) dt. \tag{A.24}$$

There is a great deal to explain in the notation in Equation A.24. First, the $S[q(*)]$ implies that action depends on the trajectory $q(t)$ and not q at some particular time t. It depends on the entire trajectory between the end points. The problem is that the space of trajectories is very rich. It is too rich to map onto the real line, see Section A.2.2, and thus the action is not a function of $q(t)$ in the sense of the usual definition of a function and thus in this case S is called a functional of $q(t)$. Note also that the action is not a functional of the derivatives of $q(t)$ since these are specified by $q(t)$. Once the trajectory is selected, the action can be evaluated and is a regular function of the initial and final events coordinates as indicated and the usual rules for differentiation hold. The Lagrangian for simple mechanical systems is, as before, the kinetic minus the potential energies. For more general cases, it is whatever gives the correct dynamics when the action is minimized. The Lagrangian is evaluated at each time, on the specified trajectory, and thus through its dependence on its arguments is a function t.

The natural trajectory is the trajectory that minimizes the action. We assume that there is a natural trajectory, $\bar{q}(t)$, and that the minimum that the action obtains is a soft one. That is the action of the trajectories near the natural trajectory differs from the action of the natural trajectory only by terms that are linear in the differences of the trajectories. Consider the differences in the actions for the natural trajectory, $\bar{q}(t)$, and a "nearby" one, $q(t) = \bar{q}(t) + \delta q(t))$ going through the initial and final events.

$$
\begin{aligned}
\delta S &= S[\bar{q}(*)](q_0, t_0; q_f, t_f) - S[\bar{q}(*) + \delta q(*)](q_0, t_0; q_f, t_f) \\
&= \int_{t_0}^{t_f} dt \left\{ L\left(\dot{\bar{q}}(t) + \delta\dot{q}(t), \bar{q}(t) + \delta q(t); t\right) - L\left(\dot{\bar{q}}(t), \bar{q}(t); t\right) \right\} \\
&= \int_{t_0}^{t_f} dt \left\{ \frac{\partial L}{\partial \dot{q}} \Big|_{\bar{q}(t)} \delta\dot{q}(t) + \frac{\partial L}{\partial q} \Big|_{\bar{q}(t)} \delta q(t) \right\} \\
&= \int_{t_0}^{t_f} dt \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \Big|_{\bar{q}(t)} \delta q(t) \right) - \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \Big|_{\bar{q}(t)} \right) \delta q(t) \right) + \frac{\partial L}{\partial q} \Big|_{\bar{q}(t)} \delta q(t) \right\} \\
&= \left(\frac{\partial L}{\partial \dot{q}} \Big|_{\bar{q}(t)} \right) \delta q(t) \Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} dt\, \delta q(t) \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \Big|_{\bar{q}(t)} \right) - \frac{\partial L}{\partial q} \Big|_{\bar{q}(t)} \right\} \\
&= 0 \quad \Rightarrow \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \Big|_{\bar{q}(t)} \right) - \frac{\partial L}{\partial q} \Big|_{\bar{q}(t)} \right\} = 0. \qquad \text{(A.25)}
\end{aligned}
$$

The development of Equation A.25 is as follows: Line one is the definition of the change in the action over the two trajectories. The next line substitutes the appropriate Lagrangians. The third line expands the first Lagrangian in a Taylor series about $\bar{q}(t)$ and subtracts the original Lagrangian. The fourth line uses $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\big|_{\bar{q}(t)} \delta q(t)\right) = \frac{\partial L}{\partial \dot{q}}\big|_{\bar{q}(t)} (\dot{\delta q}(t)) + \left(\dot{\frac{\partial L}{\partial \dot{q}}\big|_{\bar{q}(t)}}\right)\delta q(t)$ and the fact that $(\dot{\delta q}(t)) = \delta\dot{q}(t)$, since the trajectories are compared at equal times, to eliminate the $\delta\dot{q}(t)$. This is an important step which is at the heart of the derivation. $\delta\dot{q}(t)$ is not the independent variation. It follows from the $\delta q(t)$ and we need to isolate the independent variations. Using this identity, in the next line, we perform the integration of the total time derivative and leave the remaining integral in the form of $\int_{t_0}^{t_f} dt\, \delta q(t) \{\cdots\}$. The first term vanishes because we require all the trajectories to come to the same end points and thus $\delta q(t_0) = \delta q(t_f) = 0$. Requiring δS to vanish even though $\delta q(t)$ is small but not zero and arbitrary requires $\{\cdots\} = 0$ for all times which yields the Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\,|_{\bar{q}(t)}\right) - \frac{\partial L}{\partial q}\,|_{\bar{q}(t)} = 0, \tag{A.26}$$

for the dynamic. For the case of classical mechanics, when

$$L = \text{KineticEnergy} - \text{PotentialEnergy},$$

we recover the usual local dynamic. In the general case, this equation is a second order differential equation. Its solution depends on two boundary conditions usually an initial position and an initial velocity. From the Euler-Lagrange equation, the second derivatives and then all subsequent derivatives can be determined and thus the natural trajectory, $\bar{q}(t)$ is determined.

Equation A.26 is general result for functionals. The linear dependence of the functional on small changes in trajectories between given start and finish events allows the notation $\frac{\delta S}{\delta q(t)} = \{\cdots\}$ which in our case is

$$\frac{\delta S[q(*)]}{\delta q(t)}\,(q_0, t_0 : q_f, t_f)\,\big|_{\bar{q}(t)} = \left\{-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\,|_{\bar{q}(t)}\right) + \frac{\partial L}{\partial q}\,|_{\bar{q}(t)}\right\}. \tag{A.27}$$

It is important to always keep in mind that the integral form of this "derivative" is the penultimate line of Equation A.25. Note also that $\frac{\delta S[q(*)]}{\delta q(t)}\,\big|_{\bar{q}(t)}$ is time dependent even though $S[q(*)]$ is not. It is also important to note an important change in tone when going from the least action condition which involves the total trajectory to the Euler-Lagrange equation which is local in space and time. This sharp difference in interpretation comes about because the space of trajectories "nearby" to the natural trajectory can be spanned by differences that are localized on the trajectory; the nearby trajectory to the natural trajectory is the same as the natural trajectory except at some particular time on the natural trajectory there is a large deviation of place such that the time integral over the time interval is still finite.

In summary, Equation A.27 gives meaning to the idea of a derivative of a functional. For the functional[2],

$$S[q(*)](q_0, t_0; , q_f, t_f) \equiv \int_{t_0}^{t_f} L\left(\dot{q}(t), q(t), t\right) dt$$

$$\frac{\delta S[q(*)]}{\delta q(t)} = \left\{ -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} \right\}. \tag{A.29}$$

which is defined for all trajectories connecting the initial, (q_0, t_0), and final events, (q_f, t_f). The condition for the naturally occurring trajectory is that $\frac{\delta S[q(*)]}{\delta q(t)} \left.\right|_{\bar{q}(t)} = 0$

A.2.9 Hamiltonian Formulation of Mechanics

There is another important approach to dynamics beside the Euler-Lagrange equation due to Hamilton. If we identify a momentum called the conjugate momentum to $q(t)$ as

$$\bar{p}(t) \equiv \frac{\partial L}{\partial \dot{q}} \left.\right|_{\bar{q}(t)} , \tag{A.30}$$

the Euler-Lagrange becomes $\frac{dp}{dt}(t) = \frac{\partial L}{\partial q} \left.\right|_{\bar{q}(t)}$ which when the usual forms for the Lagrangian are used recovers the usual form of Newton's Laws.

Hamilton formulated an action principle that takes advantage of this substitution to develop a form of the Principle of Least Action that leads to evolution equations that have only first order in time dynamics. As opposed to the comments in the previous paragraph in which the conjugate momentum is a derivative concept, his approach treats the coordinate and the conjugate momentum on equal footing. Instead of discussing the trajectory in configuration space, Hamilton places the problem on a phase space, (q, p). The system evolves in both $q(t)$ and $p(t)$ independently but for the natural trajectory not only must you recover the correct evolution but also conditions such as Equation A.30.

[2]The more general case is that for the functional

$$S[q(*)](\cdots \dot{q}_0, q_0, t_0; \cdots \dot{q}_f, q_f, t_f) \equiv \int_{t_0}^{t_f} L\left(\cdots \ddot{q}(t), \dot{q}(t), q(t), t\right) dt$$

the functional derivative is

$$\frac{\delta S[q(*)]}{\delta q(t)} = \left\{ \cdots + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} \right\}. \tag{A.28}$$

Equation A.30 written in general terms, $p(t) \equiv \frac{\partial L}{\partial \dot{q}}(\dot{q}(t), q(t); t)$ can be inverted to solve for $\dot{q}(t)$. The existence of this solution is the same condition that you get when you solve for the $\ddot{q}(t)$ from the Euler-Lagrange equation.

Writing the Lagrangian as

$$L(\dot{q}(p(t), q(t), t), q(t)t) = p(t)\dot{q}(p(t), q(t), t) - H(p(t), q(t), t), \qquad \text{(A.31)}$$

we can find the change in the action for independent variations in $q(t)$ and $p(t)$ using the Lagrangian in Equation A.31. Also remember that regardless of the behavior of $p(t)$, $\dot{q}(t) = \frac{d}{dt}q(t)$ and that, therefore, $\delta\dot{q} = \delta\frac{dq}{dt} = \frac{d}{dt}\delta q(t)$ since the vatiations are taken at the same time, see Figure A.14.

$$\begin{aligned}
\delta\mathcal{S} &= \int_{t_0}^{t_f} dt\delta\left\{p(t)\dot{q}(t) - H(p(t), q(t), t)\right\} \\
&= \int_{t_0}^{t_f} dt\left\{p(t)\delta\dot{q}(t) + \dot{q}(t)\delta p(t) - \delta p(t)\frac{\partial H}{\partial p} - \delta q\frac{\partial H}{\partial q}\right\} \\
&= \int_{t_0}^{t_f} dt\left\{\frac{d}{dt}(p(t)\delta q(t)) - \dot{p}(t)\delta q(t) + \dot{q}(t)\delta p(t) - \delta p\frac{\partial H}{\partial p} - \delta q\frac{\partial H}{\partial q}\right\} \\
&= p(t)\delta q(t)\Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} dt\left\{\delta q(t)\left(\dot{p}(t) + \frac{\partial H}{\partial q}\right) - \delta p(t)\left(\dot{q}(t) - \frac{\partial H}{\partial p}\right)\right\} \\
&= 0. \qquad \text{(A.32)}
\end{aligned}$$

Again the first term vanishes because $\delta q(t_f) = \delta q(t_o) = 0$. The requirement of the stationarity of the action for arbitrart variations of $q(t)$ and $p(t)$ for the naturally occurring trajectory implies

$$0 = \dot{\bar{p}}(t) + \frac{\partial H}{\partial q}\Big|_{\bar{q}(t)\bar{p}(t)} \qquad \text{(A.33)}$$

$$0 = \dot{\bar{q}}(t) - \frac{\partial H}{\partial p}\Big|_{\bar{q}(t)\bar{p}(t)} \qquad \text{(A.34)}$$

These equations are the Hamiltonian dynamical equations.

In Hamiltonian mechanics all issues involving the status of the system are determined by functions of the $p(t)$ and $q(t)$. This is not really different then in the Euler-Lagrange approach which leads to a second order in time evolution and thus in which it is required to know the initial position and velocity.

A.2.10 Schwinger Variational Principle

The Schwinger Variational Principle extends the usual least action articulation of dynamics in two ways. First, it allows variations of the test trajectories at the end points and at different times. This is a necessary addition if we are to make sense of transformation of systems with equivalent dynamics, symmetry, see Section A.3. In addition, Schwinger extends the process of introducing new elements which introduce constraints. The extension of the

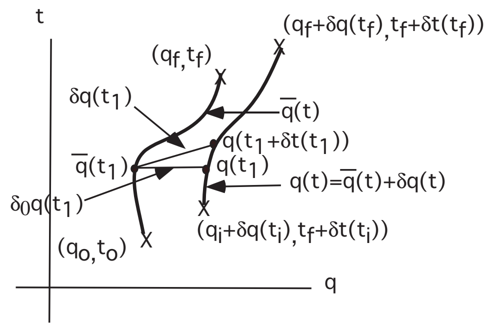

Figure A.15: **Schwinger Variation with End Point and Time Shift** In Schwinger variation, the entire trajectory including the end points and the time of comparison are shifted. The condition on the natural trajectory is that the action for a “nearby” trajectory, $q(t') = \bar{q}(t) + \delta q\,(t + \delta t(t))$, should then produce an action that differs from the natural trajectory action by a small amount in the form that the action $\Delta S = \frac{\delta S[q(*)]}{q(t)}|_{\bar{q}(t)}\delta q(t) + G_2(t_f) - G_1(t_i)$ where the functions $G_i(t)$ are independent of the trajectories depending only on the endpoint values. The difference between the action of this nearby trajectory and that of the action on the natural trajectory should be small and proportional to the differences between the natural and nearby trajectory, $\delta q(t + \delta t) = \delta_0 q(t) + \frac{dq}{dt}(t)\delta t(t)$ where $\delta_0 q(t) = q(t) - \bar{q}(t)$. Since this difference in the actions must vanish for all possible small nearby trajectories the coefficient of the difference in the trajectories must be zero.

A.3 The Nature of Symmetry in Physics

In many respects, symmetry in physics is very similar to that in art; there are families of transformations that lead to unimportant changes in the situation. The differences deal with the things on which the transformations act and the definition of unimportant. As expected, in addition, the language that described the actions are more precise and abstract. We will also categorize the transformations of physics in a formal way and use these labels to describe important results.

A.3.1 Discrete Transformations

These are changes that can only be applied in discrete steps. Bilateral or mirror symmetry about a plane is an example from art. For the snow flakes, the rotations at $\theta = n\frac{\pi}{3}$ for $n = 1, 2....$ is an example of a family of discrete transformations that produce a symmetry. What do you think happens for $n = 0$? Is this the same as $n = 6$? The rule is that, once you have a set of transformations, the set must contain all combinations of the transformations for the set to be complete.

The example in physics that corresponds to bilateral symmetry is called a spatial inversion which is to replace places in one directions by their opposite. In a world with on space dimension, replace x by $-x$. In a world with three spatial directions, replace (x, y, z) with $(-x, y, z)$. This is like placing a mirror in the plane $y = 0, z = 0$. This is obviously a discrete transformation. You also note that, if it is applied twice, there is no change. It is said to be a discrete transformation of cycle two; it has two elements, do nothing, the identity transformation, and the inversion. There are many discrete transformations of cycle two: if you have identical particles, you can interchange the particles, you can invert the time, you can do a spatial inversion along the y or z axis, ...

There are, of course, discrete transformations with cycles higher than two. The snowflake example from art carries over to physics. Rotations about the origin by an angle of $\frac{2\pi}{n}$ is an example of a discrete transformation with n cycles.

You can also have a family of discrete transformations that have an infinite number of elements. In one spatial dimension, you can shift the origin by a fixed amount, a. You can do this any number of times generating a set of transformations that has a countable infinite number of members.

It is important to realize that the method by which the members of a family of discrete transformations are labeled must itself be a discrete set of labels and that the members of a discrete set of transformations cannot be labeled by a continuous variable.

A.3.2 Continuous Transformations

Continuous transformations are changes that can be applied for arbitrarily small changes. The labeling of the transformations is a continuous parameter. Rotations about a point are a valuable example. In art, a world of concentric rings would enjoy a symmetry for rotations about the center point. These changes in angle can take any value from zero to 2π. This idea is carried over to physics. In a three dimensional space, rotations about an axis are a family of transformations. These transformations are an example of continuous transformations. Other obvious examples are translations in space and time. Changes in the scale of length discussed in Sections **??**, and **??** is also a continuous set of transformations. **Again it is important to realize that a continuous family of transformations can only be labeled by a continuous variable.**

It is possible to make a discrete family of transformations from subsets of continuous transformations such as the set of rotations used in the snowflake example of Figure **??** in Section **??**. Of course, the reverse process is not possible; you cannot make a continuous family of transformations from a subset of a discrete family no matter how large the set of discrete transformations.

A.3.3 Identity Transformation

The identity transformation is the one that leaves everything alone. The example $n = 0$ in the discrete case above is an identity transformation. Note that $n = 6m$ where $m = 1, 2, 3...$ is also the identity and we already had it in the set of transformations. In fact, any transformation in which $n > 6$ is the same as the transformation $n' = \mathrm{mod}_6(n)$.

A.3.4 Examples of symmetry in situations like physics

You are planning a trip between Austin and College Station. There are several routes.

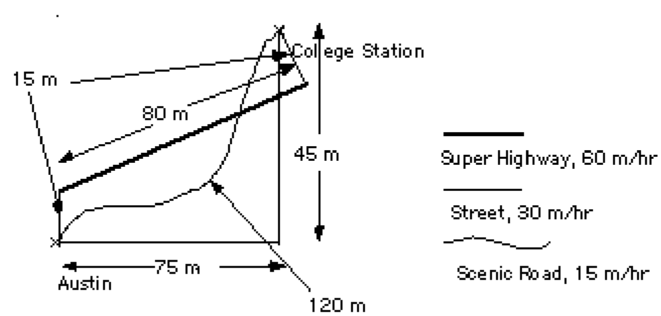

Figure A.16: **Paths to Texas A&M** Several routes for a trip between UT in Austin and Texas A & M in College Station.

A.3.5 Physics transformations:

There are several criteria that you can use to select the route: least time, least distance, see most trees and hills - one hill is worth a dozen trees. There are several changes that you can make in the system: interchange Austin and College Station, interchange super highways and streets, make the speed limit 50 $\frac{\mathrm{m}}{\mathrm{hr}}$, measure all distances in feet. These are all discrete transformations. You could shift the entire thing a distance x to the east and we all know that as you go east there are no longer any hills. You could shift all the distances by a scaling factor α. These are continuous transformations. For all of these you can see if the transformation effects the evaluation of the criteria.

From this example you see that you need both a set of transformations and a criteria.

A.4 Examples of Symmetry in physics

In physics we are interested in what happens to things in space time, i. e. events. These are labeled by (x,t). An event is a point in a space time diagram. A connected set of events is a trajectory. This is the path that a particle follows as it moves. This is often called a particles world line.

In physical systems, we can either change the events in the transformation process or change the measuring system that is used to identify the events. The former case is called the active view of transformations and the latter is the passive view. Obviously, they are equivalent descriptions of the effects of the transformations and which is being used is chosen by the context of the problem.

A.4.1 Physics transformations:

Space Reflection:

This is the transformation that corresponds to the bilateral transformation that we discussed earlier. We reflect all the events through the line $x = 0$ better known of as the t axis.

$$x \to x' = -x \tag{A.35}$$

This is an example of showing this transformation in the active view. The passive view would be to have the events stay the same but reverse the direction of the positive x axis. Obviously, they are equivalent views.

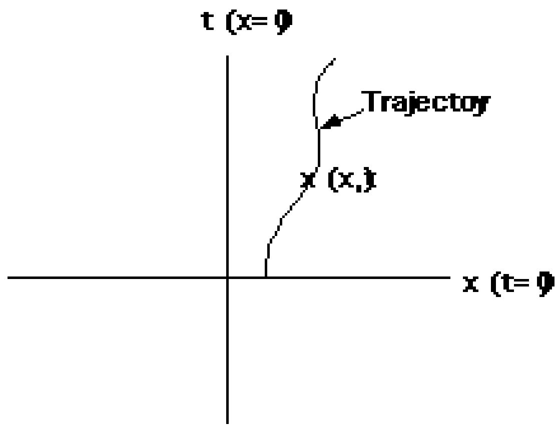

Figure A.17: **Action trajectory Trajectory 2**

Space Translation:

Shift the origin of the coordinate system.

$$x \rightarrow x' = x + a \tag{A.36}$$

Time Translation:

Shift the start of the time.

$$t \rightarrow t' = t + a \tag{A.37}$$

To be a symmetry we will require that the physics before and after the shift is the same. I have not carefully defined what I mean by "the same." I will do so shortly.

Newton's Action at a Distance Law of Gravitation

The law of force that describes the gravitational influence of one body, say body 2, on another body, say body 1, is

$$\vec{F}_{1,2} = G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3} \times (\vec{r}_2 - \vec{r}_1) \tag{A.38}$$

Similarly, the gravitational force of body 1 on body 2 can be found by interchanging the labels of particles 1 and 2.

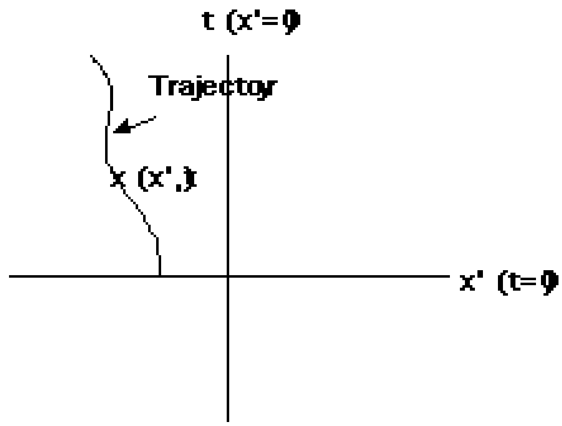

Figure A.18: **Space Reflection** Space Reflection

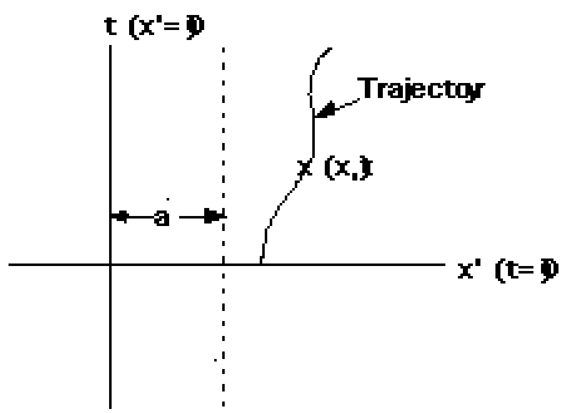

Figure A.19: **Space Translation** Space Translation

$$\vec{F}_{2,1} = G\frac{m_2 m_1}{|\vec{r}_1 - \vec{r}_2|^3} \times (\vec{r}_1 - \vec{r}_2) \tag{A.39}$$

Thus if you are operating at the level of the forces you have that if you interchange particles 1 and 2, i. e. change the labels 1 and 2, $1 \leftrightarrow 2$ and get $\vec{F}_{1,2} \rightarrow -\vec{F}_{2,1}$ This is a discrete transformation. If for some reason you are interested in the forces, this is not a symmetry. It is actually a manifestation of the Law of Action Reaction. In other words, we construct the Law of Gravitation so that it obeys the Law of Action Reaction. On the other hand, if you look at the entire set of equations without the forces, there is no change.

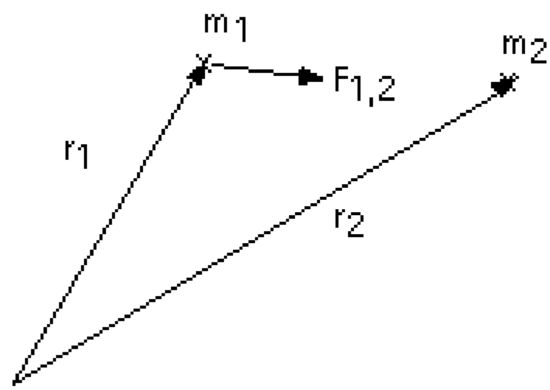

Figure A.20: **Gravitational Symmetry** Gravitational Symmetry

$$m_1\vec{a}_1 = G\frac{m_1 m_2}{|\vec{r}_2-\vec{r}_1|^3}\times(\vec{r}_2-\vec{r}_1) \tag{A.40}$$

$$m_2\vec{a}_2 = G\frac{m_2 m_1}{|\vec{r}_1-\vec{r}_2|^3}\times(\vec{r}_1-\vec{r}_2) \tag{A.41}$$

Some symmetries of this law:

This is then a symmetry. When you put a shift to all the positions by some amount, $\vec{a}$, nothing changes, i. e. $\vec{r}_i \to \vec{r}_i + \vec{a}$. This is a continuous symmetry. When you replace all the positions with the reverse position, $\vec{r}_i \to -\vec{r}_i$ again nothing changes. Remember $\vec{a}_i \to -\vec{a}_i$. This is a discrete symmetry. If you change all the distances in the problem by a scale $\vec{r}_i \to \vec{r'}_i = \lambda\vec{r}_i$, then this is not a symmetry. But, if you also change the time scale by $t \to t' = \lambda^{\frac{3}{2}}t$, then you have a symmetry. This is a continuous symmetry. Note that the identity transformation is $\lambda = 1$.

A.5 Symmetry and Action

A.5.1 Introduction

You can have the situation that you make the change and the action does not change at all. Said more carefully, you have transformed end points and transformed paths and you get the same value for the action.

Consider the free particle and translations in space.

$$\begin{aligned} x' &= x + a \\ t' &= t \end{aligned} \tag{A.42}$$

This implies that $v' = v$. Thus

$$\begin{aligned} S'(x'_f, t'_f, x'_0, t'_0; path') &= \sum_{path', (x'_0, t'_0)}^{(x'_f, t'_f)} (m\frac{v'^2}{2})\Delta t \\ &= \sum_{path, (x_0, t_0)}^{(x_f, t_f)} (m\frac{v^2}{2})\Delta t \\ &= S(x_f, t_f, x_0, t_0; path) \end{aligned} \tag{A.43}$$

If action is the basis of all physics, then we have a natural definition of a symmetry of a physical system. A physical system has a symmetry if there is a way to modify the system and yet there is no significant change in the action. It is important to be careful about the meaning of significant in this sentence. For most purposes the value of the action is not important. The action primary role is to select a path from the infinity of possibilities. In this sense, we can as a first step assert that the system is symmetric if the system before and after the change still selects the same path as the natural path. You again have to be careful because the same path is actually the same path as seen in the modified system. An example might help clarify this.

Harmonic Oscillator and Symmetry

The harmonic oscillator is one of the most important physical systems. We will discuss the physics of this system in greater detail in a later section, Section **??**, but for now will use it as another example in which to examine the role of symmetry in a physical system. For now just think of of it as a physical system that goes back and forth.

The Lagrangian for the harmonic oscillator is

$$L(v, x) = KE - PE = m\frac{v^2}{2} - k\frac{x^2}{2} \tag{A.44}$$

where k is the spring constant and m is the mass and both are given constants and have the dimension $k \overset{\text{dim}}{=} \frac{\text{Mass}}{\text{Time}^2}$ and, of course, m is a mass. Note that, if these are the only two dimensional constants that are available, then you cannot make a length but you can make a time. If you rescale the distances by an amount λ, as follows:

$$\begin{aligned} x &\rightarrow x' = \lambda x \\ t &\rightarrow t' = t \end{aligned} \quad \text{(A.45)}$$

which implies that

$$v \rightarrow v' = \frac{\Delta x'}{\Delta t'} = \lambda \frac{\Delta x}{\Delta t} = \lambda v \quad \text{(A.46)}$$

The Lagrangian for the new system is

$$L'(v', x') = KE' - PE' = m\frac{v'^2}{2} - k\frac{x'^2}{2} = m\lambda^2(\frac{v^2}{2} - k\frac{x^2}{2}) = \lambda^2 L(v, x) \quad \text{(A.47)}$$

So that

$$\begin{aligned} S'_{Path'}(x'_0, t'_0; x'_f, t'_f) &= \sum_{path', (x'_0, t'_0)}^{(x'_f, t'_f)} (m\frac{v'^2}{2} - k\frac{x'^2}{2})\Delta t' \\ &= \lambda^2 \sum_{path, (x_0, t_0)}^{(x_f, t_f)} (m\frac{v^2}{2} - k\frac{x^2}{2})\Delta t \\ &= \lambda^2 S_{Path}(x_0, t_0; x_f, t_f) \end{aligned} \quad \text{(A.48)}$$

where Path' is the Path that is at the rescaled distances

$$x'(t') = \lambda x(t) \quad \text{(A.49)}$$

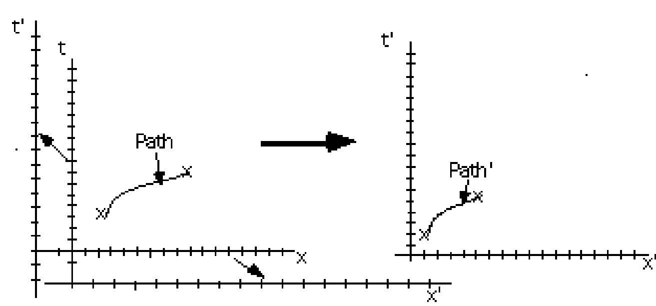

Figure A.21: **Rescale Oscillator** Rescale Oscillator

Path	Action	Path'	Action'
1	S_1	1'	$S'_{1'} = \lambda^2 S_1$
2	S_2	2'	$S'_{2'} = \lambda^2 S_2$
.	.	.	.
.	.	.	.
.	.	.	.
natural	S_{least}	natural'	$S'_{least'} = \lambda^2 S_{least}$
.	.	.	.
.	.	.	.
.	.	.	.

You get the same path even though the calculations are all different.

A.5.2 Galilean invariance

In order to show that the straight line was the solution to the free particle action problem I assumed that the action procedure was Galilean invariant and went to a special frame. The question is "is it." The action is

$$S(x_f, t_f, x_0, t_0; path) = \sum_{path, x_0, t_0}^{x_f, t_f} \Big(m\frac{v^2}{2}\Big)\Delta t \tag{A.50}$$

What happens when you make the Galilean transformation?

$$\begin{aligned} x' &= x - at \\ t' &= t \end{aligned} \tag{A.51}$$

Where a is a parameter that labels the transformations and has the dimensions of a velocity – it is actually interpreted as a velocity. With this transformation all the velocities shift, $v' = v - a$.

$$\begin{aligned} S'(x'_f, t'_f, x'_0, t'_0; path') &= \sum_{path', x'_0, t'_0}^{x'_f, t'_f} \Big(m\frac{v'^2}{2}\Big)\Delta t \\ &= \sum_{path, x_0, t_0}^{x_f, t_f} \Big(m\frac{(v-a)^2}{2}\Big)\Delta t \\ &= \sum_{path, x_0, t_0}^{x_f, t_f} \Big(m\frac{v^2}{2}\Big)\Delta t - \sum_{path, x_0, t_0}^{x_f, t_f} (mva)\Delta t + \sum_{path, x_0, t_0}^{x_f, t_f} \Big(m\frac{a^2}{2}\Big)\Delta t \end{aligned}$$

$$= S(x_f, t_f, x_0, t_0; path) - ma \sum_{path, x_0, t_0}^{x_f, t_f} v\Delta t + \left(m\frac{a^2}{2}\right) \sum_{path, x_0, t_0}^{x_f, t_f} \Delta t$$
$$= S(x_f, t_f, x_0, t_0; path) - ma(x_f - x_0) + \left(m\frac{a^2}{2}\right)(t_f - t_0) \quad \text{(A.52)}$$

The last two terms are independent of path. Therefore the path selection process the selects the least path in S will select the transformed path in S'. The action changes under the transformation but in an unimportant way. **This is not a symmetry and there is no associated conserved quantity.**When we implement this for special relativity it will become a symmetry.

A.5.3 More on Symmetry and Action

The easiest way to guarantee that the action is symmetric under a set of transformations is to construct it only from the form invariants for that set of transformations. In fact, it is a necessary and sufficient condition that the action is symmetric that it be composed of only form invariants for that set of transformations.

As an example consider the action for a satellite of mass m in orbit around the earth. Locating the earth at the origin, the action is

$$S(\vec{x}_0, t_0, \vec{x}_f, t_f; path) = \sum_{Path, \vec{x}_0, t_0}^{\vec{x}_f, t_f} (m\frac{\vec{v}^2}{2} + Gm\frac{M_{earth}}{r})\Delta t \quad \text{(A.53)}$$

This action is composed of $\vec{v}^2$ which is a form invariant for rotations about the origin. r is the distance from the origin and it is also a form invariant for rotations. Obviously Δt is a form invariant for rotations. Thus this action has a symmetry that is the set of transformations that are the rotations about the origin.

A.5.4 Noether's Theorem

For every continuous transformation that is connected to the identity that is a symmetry, no important change, there is a conserved quantity. Noether's Theorem also tells you how to construct the conserved quantity. When I tell you what the question is and thus when a change is important, I can tell you how to construct the conserved quantity.

Space translation Symmetry

The conserved quantity that is associated with situations with space translation symmetry is called linear momentum. In certain cases it is $\vec{p} = m\vec{v}$ but not all the time. I will tell you when those cases are.

Rotation translation symmetry

The conserved quantity that is associated with situations with space rotation symmetry is called angular momentum. Rotations are a vector quantity. Again in certain cases it is $\vec{L} = m\vec{r} \times \vec{v}$.

Time translation Symmetry

The conserved quantity that is associated with situations with time translation symmetry is called energy. This is actually the case all the time but the form of the energy may change.

Galilean Invariance

This is almost a symmetry classically and becomes a full blown symmetry in the modern language. First, let's discuss what the transformation is.

There is no experiment that can be performed that can measure the velocity of an moving observer. We can detect the presence of accelerations and measure the relative velocity between two bodies but we cannot measure absolute velocities.

Another way to say the same thing is that, if you are not accelerating, you are always at rest in your own rest frame.

In the language of transformations, all the laws of physics must be invariant under a transformation of the form

$$\begin{aligned} \vec{x} &\rightarrow \vec{x}' = \vec{x} + \vec{R} + \vec{v}t \\ t &\rightarrow t' = t \end{aligned} \tag{A.54}$$

where $\vec{R}$ and $\vec{v}$ are constants that are the parameters that label the continuous transformations. They can be interpreted in terms of two coordinate systems this can be interpreted as the difference in the measurements of two relatively displaced and relatively moving coordinate systems.

Although this is a continuous symmetry that is connected with the identity, it is not a symmetry classically. I will explain this later. Since this is

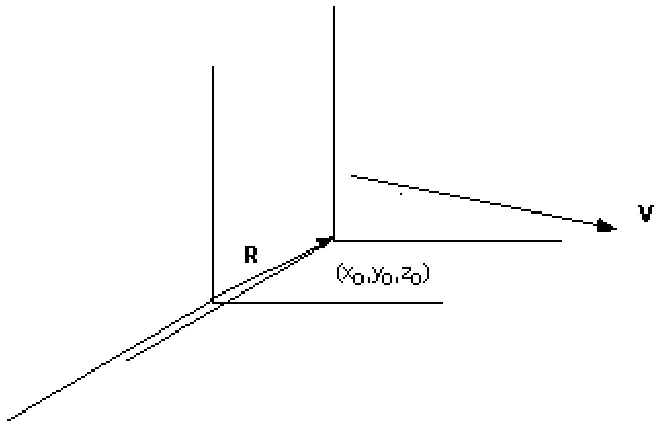

Figure A.22: **Galilean Invariance** Galilean Invariance

not a symmetry, there is no conserved quantity that is the result of Galilean invariance in classical physics.

You should apply this transformation to the gravitational force above and see that the neither the forces nor the equations change. If you use these as your criteria for a symmetry, this would be a symmetry. It is not so we see that we need a better criteria.

Add some notes on the two observers moving by each other.

Please read the Feynman lecture. I do not expect that all of you will follow this material. It is a basis for Noether's Theorem.

Consider a change in the system that also changes the description of initial and final events. This is what will generally happen. Here, when you do the transformations, you will get in addition to the usual terms of the integral of the Lagrangian but also terms from the end points. Our modified form of Feynman's equation

$$\begin{aligned} \delta S &= \left(\left(\frac{\delta L}{\delta v}\mid_{x_{nat}(t)}\right)\delta x\right)_{t_f} - \left(\left(\frac{\delta L}{\delta v}\mid_{x_{nat}(t)}\right)\delta x\right)_{t_0} \\ &\quad + \int_{t_0}^{t_f}\left(\frac{d}{dt}\left(\frac{\delta L}{\delta v}\right) - \frac{\delta L}{\delta x}\right)\delta x dt \end{aligned} \tag{A.55}$$

To get the action to be stationary now we will require that as before the integrand vanish

$$\frac{d}{dt}\left(\frac{\delta L}{\delta v}\right) - \frac{\delta L}{\delta x} = 0 \tag{A.56}$$

but also that the terms from the end points vanish. This part simply selects the natural path. To understand the end points consider an example, the simple translation. In this case δx is simply a number that is added to all points in the path.

$$\delta x(t_f) = \delta x(t_0) = a \tag{A.57}$$

or

$$\left(\left(\frac{\delta L}{\delta v} \mid_{x_{nat}(t)}\right) \delta x\right)_{t_f} - \left(\left(\frac{\delta L}{\delta v} \mid_{x_{nat}(t)}\right) \delta x\right)_{t_0} = \left(\left(\frac{\delta L}{\delta v} \mid_{x_{nat}(t)}\right)_{t_f} - \left(\frac{\delta L}{\delta v} \mid_{x_{nat}(t)}\right)_{t_f}\right) a \tag{A.58}$$

Setting this to zero, yields

$$\left(\frac{\delta L}{\delta v} \mid_{x_{nat}(t)}\right)_{t_f} = \left(\frac{\delta L}{\delta v} \mid_{x_{nat}(t)}\right)_{t_f} \tag{A.59}$$

But $\frac{\delta L}{\delta v} \mid_{x_{nat}(t)}$ is what you would **define** as the momentum. It is the momentum when you use the usual Lagrangian. Thus this is nothing more than the statement that momentum is conserved.

$$p(t_f) = p(t_0) \tag{A.60}$$

This is a special case of a general theorem called Noether's Theorem. Given any transformation that can be connected with the identity transformation, no change, by a continuous parameter. There will always be a conserved quantity. In the above example the transformation is translation. In the limit $a \to 0$ you have no translation and thus no change and the identity transformation. In this case, the conserved quantity is the linear momentum.

Another way of looking at this result is that, once you have selected the natural path and if you include the end point variations, the action is a function of the end points only. If the symmetry transformation changes the end points you have

$$\delta S_{Nat}(x_0, t_0; x_f, t_f) = \frac{\delta S_{Nat}}{\delta x_0}\delta x_0 + \frac{\delta S_{Nat}}{\delta x_f}\delta x_f + \frac{\delta S_{Nat}}{\delta t_0}\delta t_0 + \frac{\delta S_{Nat}}{\delta t_f}\delta t_f \tag{A.61}$$

In the case of translations,

$$\delta x(t_f) = \delta x(t_0) = a \tag{A.62}$$

and all the δt_i are zero.

Thus we get

$$\frac{\delta S}{\delta x_f} = -\frac{\delta S}{\delta x_0} = p = \text{constant} \tag{A.63}$$

An Example

For the free particle,

$$S_{natural} = m\frac{(x_f - x_0)^2}{2(t_f - t_0)} \tag{A.64}$$

$$p = \frac{\delta S}{\delta x_f} = m\frac{(x_f - x_0)}{(t_f - t_0)} = mv \tag{A.65}$$

since v is a constant.

We noted above that the satellite in orbit is a case that is invariant under rotations about the origin. This set of transformations is a continuous set and thus there is a conserved quantity. In this case we call it the angular momentum. The construction of this conserved quantity involves cumbersome notation because it only makes sense in a system with at least two spatial dimensions and thus involves vector notation. In addition, it is computationally difficult to find an expression for the natural path. But note that the free particle Lagrangian is also composed only of form invariants for rotations about the origin. Thus this set of transformations is also a symmetry for this case. The analysis is still cumbersome because of the vector notation. I am aware that you will not be able to reproduce this analysis. All that I ask is that you follow it.

We will work in two spatial dimensions. For this case the action is

$$S(\vec{x}_0, \vec{t}_0; \vec{x}_f, \vec{t}_f) = \sum_{\text{NaturalPath}, \vec{x}_0, \vec{t}_0}^{\vec{x}_f, \vec{t}_f} m\frac{\vec{v}^2}{2}\Delta t \tag{A.66}$$

and as we see is composed of only form invariants not only of translations in space and time but also for rotations. The quantity $\vec{v}^2$ is invariant under rotations.

For the natural path the action is

$$S_{natural} = m\frac{(\vec{x}_f - \vec{x}_0)^2}{2(t_f - t_0)} \tag{A.67}$$

and the change in the action caused by the end point changes are

$$\delta S_{Nat}(\vec{x}_0, t_0; \vec{x}_f, t_f) = \frac{\delta S_{Nat}}{\delta \vec{x}_0} \cdot \delta \vec{x}_0 + \frac{\delta S_{Nat}}{\delta \vec{x}_f} \cdot \delta \vec{x}_f + \frac{\delta S_{Nat}}{\delta t_0} \delta t_0 + \frac{\delta S_{Nat}}{\delta t_f} \delta t_f \tag{A.68}$$

For rotations, δt_0 and δt_f are zero. The $\delta \vec{x}_0$ and $\delta \vec{x}_f$ are the displacements of the end points that result from the rotation. For a rotation through an angle θ, they are

$$\delta \vec{x}_0 \tag{A.69}$$

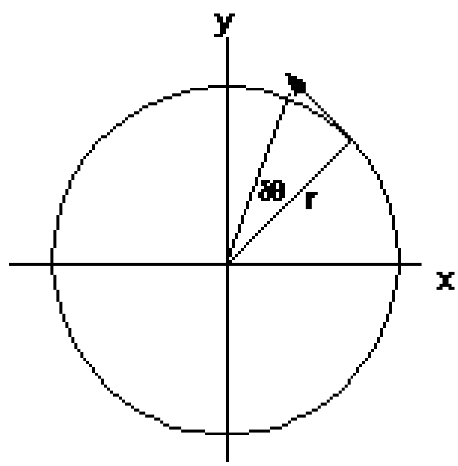

Figure A.23: **Rotation** Rotation.

From the rule above we need the change in the S_{Nat} along this direction.

As in the translation example we see that the change in S with changes in position is the regular momentum. Thus the thing that multiplies $\delta\theta$ in the change in action is the momentum along this direction times the distance. This is what we always called the angular momentum.

Thus we get the rather complicated object

$$L_{axis} = \frac{\delta S_{Nat}}{\delta \vec{x}_0} \cdot r_0 \left(\theta\right)_0 \tag{A.70}$$

The lesson of all this is that the symmetry implies that there is a conserved quantity. These are the things that we call momenta or energy etc. The form that they take depends on the nature of the Lagrangian.