

Appendix C

Propagators and Green's Functions

C.1 Introduction

The technique of solution for partial differential equations by means of integrals over propagators and Green's functions is the most powerful and useful one available. Its flexibility and generalizability make it useful in almost all applications of field theory. The basis for its application is Green's integral theorems. The second theorem

$$\int_V \left\{ \psi \vec{\nabla}^2 \phi - (\vec{\nabla}^2 \psi) \phi \right\} dV = \oint_{\partial V} \left\{ \psi \frac{\partial \phi}{\partial n} - \left(\frac{\partial \psi}{\partial n} \right) \phi \right\} dS \quad (\text{C.1})$$

which was originally developed by Green for the solution to problems of flow and electricity and magnetism [Green 1828]. The theorem is easily developed from Gauss's theorem. In this section, we will apply an offshoot of this technique to the simplest case of a field theory in $(0,1)$ dimensions; that is an ordinary differential equations of functions of one variable. This simple example will allow us to set some of the terminology and provide a starting point to the analysis of the n -spatial dimensional cases which will be analyzed in the subsequent sections.

The problem at hand for the $(0,1)$ ¹ field is the ordinary differential

¹This is a standard notation for the size of a space-time. Our space-time being $(3,1)$, three Euclidean dimensions and one time. In this case, three coordinate intervals whose difference contributes to the metric with the same sign and one with the opposite sign, $ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$.

equation

$$\frac{d^2\phi}{dt^2} = f(t) \quad (\text{C.2})$$

with boundary conditions. This example is not as trivial as it appears but is one whose solution we all know. It is the case of a time dependent force $f(t)$ acting on a unit mass. It is also the Poisson equation in one spatial dimension, a subject of the next section.

This problem is a special case of the more general case of the Sturm-Liouville system. Here the differential is

$$-\left(\frac{d^2}{dx^2} + q(x)\frac{d}{dx} + r(x)\right)\phi(x) \equiv L\phi(x) \quad (\text{C.3})$$

and the homogeneous Sturm-Liouville equation is $L\phi = 0$ and the inhomogeneous equation is $L\phi = f(x)$. Consider two solutions to the homogeneous Sturm-Liouville equation, $\phi_1(x)$ and $\phi_2(x)$. The Wronskian of these two solutions is defined as

$$W(\phi_1, \phi_2) \equiv \phi_1(x)\frac{d\phi_2}{dx}(x) - \frac{d\phi_1}{dx}(x)\phi_2(x) \equiv W(x). \quad (\text{C.4})$$

The solutions $\phi_1(x)$ and $\phi_2(x)$ are linearly dependent if $W(x) = 0$. If the two solutions are linearly independent $W(x) = \text{const} \times e^{-\int dxq(x)}$. The general form of the solution for the inhomogeneous Sturm-Liouville equation is $\phi(x) = \phi_c(x) + \phi_p(x)$ where $\phi_c(x)$ solves the homogeneous Sturm-Liouville equation and has two adjustable parameters and is called the complementary solution and $\phi_p(x)$ is *any* solution of the inhomogeneous equation and is called the particular solution.

We define a Green's function as the solution to

$$LG(x|x') = \delta(x - x') \quad (\text{C.5})$$

where the boundary conditions are chosen to suit the problem. The particular solution is then

$$\phi_p(x) = \int dx' G(x|x')f(x') \quad (\text{C.6})$$

The initial value problem (IVP) is the case in which the value of $\phi(x_1) = \alpha$ and $\frac{d\phi}{dx}(x_1) = \beta$ are given. Let $\phi_1(x)$ be a solution of the homogeneous Sturm-Liouville equation satisfying the boundary conditions and $\phi_2(x)$ be

any solution of the homogeneous Sturm-Liouville equation linearly independent of $\phi_1(x)$. Then $\phi_c(x) = \phi_1(x)$ and the Green's function is

$$G(x|x') = \theta(x - x') \frac{\{\phi_2(x')\phi_1(x) - \phi_1(x')\phi_2(x)\}}{W(x')}. \quad (\text{C.7})$$

That this form is a solution of Equation C.5 is easily seen by integrating that equation in an interval around x' . The only term which is non-zero is $\frac{dG}{dx} = \theta(x - x')^2$. This jump in the first derivative is a general observation when using Green's function techniques for second order systems. Differentiation of our construction, Equation C.7, obviously satisfies this condition.

Let's apply this general result to our particular example with initial value boundary conditions, $\phi(0) = \alpha$ and $\frac{d\phi}{dx}(0) = \beta$. The homogeneous case of our differential equation, Equation C.2, has solutions of the form $\phi(t) = At + B$. Thus our $\phi_1(t) = \phi_c(t) = \beta t + \alpha$. The solution $\phi_2(t)$ can be a constant, γ . The Wronskian is $-\beta\gamma$ which is a non-zero constant and consistent with the two solutions linear independence. The Green's function is $G(t|t') = \theta(t - t')(t' - t)$ and thus the solution is $\phi(t) = \beta t + \alpha - \int_0^\infty dt' \theta(t - t')(t' - t)f(t') = \beta t + \alpha - \int_0^t dt'(t' - t)f(t')$. Let's check that this solution does satisfy Equation C.2 and the IVP boundary conditions. Obviously, $\phi(0) = \alpha$. Since $\frac{d\phi}{dt}(t) = \beta + \int_0^t dt' f(t')$, $\frac{d\phi}{dt}(0) = \beta$. Also $\frac{d^2\phi}{dt^2}(t) = f(t)$ as required.

For boundary value problems, the unknown function or its first derivative is known at two points, for example $\phi(x_1) = \alpha$ and $\phi(x_2) = \beta$, and the problem is to find the form of the solution in the interval. This problem is the one dimensional version of the problem that Green originally addressed for the two dimensional case. First consider the case of homogeneous boundary conditions. These can be Dirichlet (DBC), $\phi(x_1) = \phi(x_2) = 0$, or Neumann (NBC), $\frac{d\phi}{dx}(x_1) = \frac{d\phi}{dx}(x_2) = 0$, or mixed. The procedure is similar to the previous analysis. For the case of DBC, choose $\phi_1(x)$ to satisfy the homogeneous Sturm-Liouville equation with boundary condition $\phi_1(x_1) = 0$. Choose $\phi_2(x)$ to satisfy the homogeneous Sturm-Liouville equation with boundary condition $\phi_2(x_2) = 0$. Thus, $\phi_1(x)$ and $\phi_2(x)$ will be linearly independent. The unique solution to the inhomogeneous Sturm-Liouville problem is the particular solution, Equation C.6, with

$$G(x|x') = - \frac{\{\theta(x' - x)\phi_2(x')\phi_1(x) + \theta(x - x')\phi_1(x')\phi_2(x)\}}{W(x')}. \quad (\text{C.8})$$

The solution for the case of homogeneous NBC or mixed the solution construction is similar.

²See Appendix B for the rules for integration of generalized functions.

Applying this to our example differential equation, Equation C.2, in the interval between $t = 0$ and $t = 1$, with mixed homogeneous boundary conditions, $\phi_1(t) = \alpha t$ and $\phi_2(t) = \gamma$ and the Wronskian is $-\alpha\gamma$. The Green's function is $\theta(t' - t)t + \theta(t - t')t'$ and the solution for the inhomogeneous equation with homogeneous mixed boundary conditions is $\phi(t) = \phi_c(t) + \phi_p(t) = t \int_t^1 dt' f(t') + \int_0^t dt' t' f(t')$ since the only solution to the homogeneous Sturm-Liouville equation with homogeneous boundary conditions is the trivial function $\phi_c(t) = 0$. Again, it is an easy exercise to show that $\phi(0) = 0$. Since $\frac{d\phi}{dt}(t) = \int_t^1 dt' f(t') - tf(t) + tf(t) = \int_t^1 dt' f(t')$, $\frac{d\phi}{dt}(1) = 0$. If the BVP had inhomogeneous boundary conditions, the $\phi_c(t)$ would have provided them. For example, if the boundary conditions were $\phi(0) = \alpha$ and $\frac{d\phi}{dt} = \beta$, then $\phi_c(t) = \alpha t + \beta$.

It is worthwhile to introduce some of the usual language that is associated with the use of Green's integral techniques. From the defining equation for the Green's function, Equation C.5, or the particular solution, Equation C.6, it is easy to see why the Green's function is often called the influence function. $G(t|t')$ is the response at time t of the system to a unit source at t' . Note also that from Equation C.5 that $G(t|t') = G(t'|t)$. This symmetry of the Green's function is called reciprocity. A unit source at t' produces an effect $G(t|t')$ at t and a unit source at t produces an effect $G(t'|t)$ at t' . There is also a very popular notation used in the presentation of Green's functions. Consider our example Green's function, $\theta(t' - t)t + \theta(t - t')t'$. By defining a simple function of the two variables t and t' , $t_{>} \equiv \text{Max}(t, t') = \begin{cases} t & t > t' \\ t' & t < t' \end{cases}$, the Green's function for this example can be written simply as $G(t|t') = t_{>}$. There is a corresponding form used to select the other variable, $t_{<} \equiv \text{Min}(t, t') = \begin{cases} t & t < t' \\ t' & t > t' \end{cases}$ and our simple Green's function could as well be written $G(t|t') = t_{<}$ or $G(t|t') = t'_{>}$. The most usual form for the Green's function is $G(t|t') = g(t_{>})h(t_{<})$.

Note that one essential ingredient in the construction of the Green's function is the identification of the two functions $\phi_1(t)$ and $\phi_2(t)$ that satisfy the homogeneous Sturm-Liouville equation with homogeneous boundary conditions. We require that these solutions be linearly independent, the Wronskian be non-zero. Although this does not happen in the case of our simple example, Equation C.2, it is possible with more general forms of the Sturm-Liouville equation to have solutions at the two boundaries that are not be linearly independent. If the Sturm-Liouville equation with homoge-

neous BV has zero eigenvalue³, a single non-trivial solution can be found that fits the boundary conditions.

C.1.1 Diversion on the Sturm-Liouville System

The Sturm-Liouville system was introduced above in Equation C.3. This is an important tool of analysis and a review of its structure is required. Consider the general bilinear form $B[\phi(x), \psi(x)] \equiv a(x)\frac{d\phi}{dx}(x)\frac{d\psi}{dx}(x) + b(x)\frac{d\phi}{dx}(x)\psi(x) + c(x)\phi(x)\frac{d\psi}{dx}(x) + d(x)\phi(x)\psi(x)$ for the two functions $\phi(x)$ and $\psi(x)$ defined on the interval x_0 to x_1 . Integrating B over the interval and removing all the derivatives of $\psi(x)$ by integration by parts, we obtain

$$\int_{x_0}^{x_1} B[\phi, \psi] = - \int_{x_0}^{x_1} \psi(x) L_0[\phi(x)] dx + \left(a(x) \frac{d\phi}{dx}(x) + c(x) \phi(x) \right) \psi(x) \Big|_{x_0}^{x_1}$$

where $L_0[\phi(x)] = \frac{d}{dx} \left(a(x) \frac{d\phi}{dx}(x) \right) - b(x) \frac{d\phi}{dx}(x) + c(x) \frac{d\phi}{dx}(x) + \frac{dc}{dx}(x) \phi(x) - d(x) \phi(x)$. Now integrating B over the interval and removing all the derivatives of $\phi(x)$ by integration by parts, we obtain

$$\int_{x_0}^{x_1} B[\phi, \psi] = - \int_{x_0}^{x_1} \phi(x) M_0[\psi(x)] dx + \left(a(x) \frac{d\psi}{dx}(x) + c(x) \psi(x) \right) \phi(x) \Big|_{x_0}^{x_1}$$

where $M_0[\psi(x)] = \frac{d}{dx} \left(a(x) \frac{d\psi}{dx}(x) \right) + b(x) \frac{d\psi}{dx}(x) + \frac{db}{dx}(x) \psi(x) - c(x) \frac{d\psi}{dx}(x) - d(x) \psi(x)$. Thus

$$\begin{aligned} \int_{x_0}^{x_1} \psi(x) L_0[\phi(x)] dx - \int_{x_0}^{x_1} \phi(x) M_0[\psi(x)] dx = \\ \left[a(x) \left(\frac{d\phi}{dx} \psi(x) - \frac{d\psi}{dx} \phi(x) \right) \right. \\ \left. + (c(x) - b(x)) \phi(x) \psi(x) \right] \Big|_{x_0}^{x_1} . \end{aligned}$$

The two differential operators L_0 and M_0 are defined by each other, given L_0 we know M_0 and visa versa, and they are said to be adjoints of one another. If $L_0\phi(x) = M_0\phi(x)$ for all functions in the interval, L_0 or M_0 are said to be self adjoint.

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³A homogeneous Sturm-Liouville problem has an eigenvalue if there exists solutions to the new differential equation $L\phi(x) = \lambda_n\phi(x)$ with the same boundary conditions. Obviously, if the Sturm-Liouville eigenproblem has a zero eigenvalue, $\lambda_0 = 0$, the associated eigenfunction satisfies the homogeneous Sturm-Liouville equation and the homogeneous boundary conditions.

C.2 Potential Problems

The original application of the Green's theorem was to potential problems in electromagnetism. The generic example is the Poisson equation,

$$\vec{\nabla}^2 \phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0} \quad (\text{C.9})$$

with either $\phi(\vec{x})$ given on the enclosing surface bounding the volume under study or $\frac{\partial \phi}{\partial \vec{x}}(\vec{x}) \cdot \hat{n}$ with \hat{n} the unit normal given on the bounding surface, see Section ???. Following the pattern of Section C.1, examples of the first case are called Dirichlet boundary conditions (DBC) and the second are Neumann boundary conditions (NBC).

This equation in a one dimension case, either $(0, 1)$ or $(1, 0)$, is the example problem that was discussed in the previous section, Section C.1. It is also relevant to higher number of spatial dimensions if there is symmetry in all but one dimension. The spatial symmetry implies that the charge distribution and potential function are independent of all but one of the spatial variables and thus the form of the Poisson equation becomes an ordinary differential equation and takes the form of a Sturm-Liouville problem. For example, the Poisson equation in three spatial dimensions with symmetry in the $x y$ directions is $\frac{d^2 \phi}{dz^2}(z) = -\frac{\rho(z)}{\epsilon_0}$. In the case with angular symmetry, the Poisson equation is $\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) (r) = \frac{d^2 \phi}{dr^2}(r) + \frac{2}{r} \frac{d\phi}{dr}(r) = -\frac{\rho(r)}{\epsilon_0}$. In both of these cases, the differential operator is of the Sturm-Liouville type.

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In the case of more than one dimension, the equation is elliptic, in fact in standard form, see Section 2.2.2. A direct application of Green's Second Integral Theorem, Equation C.1, and the defining equation for the construction of the Green's function,

$$-\vec{\nabla}^2 G(\vec{x} | \vec{x}') = \delta(\vec{x} - \vec{x}'), \quad (\text{C.10})$$

yields

$$\begin{aligned} \phi(\vec{x}) = & \int_V d^n \vec{x}' G(\vec{x} | \vec{x}') \frac{\rho(\vec{x}')}{\epsilon_0} \\ & + \int_{S \supset V} d^{n-1} \vec{x}' \left[G(\vec{x} | \vec{x}') \frac{\partial \phi}{\partial \vec{x}}(\vec{x}') \cdot \hat{n} \Big|_S \right. \\ & \left. - \frac{\partial G}{\partial \vec{x}'}(\vec{x} | \vec{x}') \cdot \hat{n} \phi(\vec{x}') \Big|_S \right]. \end{aligned} \quad (\text{C.11})$$

This result provides a construction for the solution given boundary conditions that are either Dirichlet which has $\phi(\vec{x})|_S$ known, or Neumann with $\frac{\partial\phi}{\partial\vec{x}}(\vec{x}) \cdot \hat{n}|_S$ known, or mixed with the understanding that the Green's function boundary conditions are homogeneous of the same type. For example, with Dirichlet boundary conditions given for $\phi(\vec{x})$, the appropriate boundary conditions for the Green's function are homogeneous Dirichlet boundary conditions, $G(\vec{x}|\vec{x}') = 0$. In this fashion, the unknown surface term is forced to vanish. Thus for the Dirichlet problem the solution is

$$\phi(\vec{x}) = \int_V d^n \vec{x}' G_D(\vec{x}|\vec{x}') \frac{\rho(\vec{x}')}{\epsilon_0} - \int_{S \supset V} d^{n-1} \vec{x}' \frac{\partial G_D}{\partial \vec{x}'}(\vec{x}|\vec{x}') \cdot \hat{n} \phi(\vec{x}')|_S \quad (\text{C.12})$$

where $G_D(\vec{x}|\vec{x}') = 0$ for \vec{x}' on the bounding surface. This solution to the Dirichlet problem is unique. Consider two solutions. Due to the linearity of the Poisson equation, the difference of two solutions is a solution, $\vec{\nabla}^2 \psi(\vec{x}) = \vec{\nabla}^2 \phi_1(\vec{x}) - \vec{\nabla}^2 \phi_2(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0} + \frac{\rho(\vec{x})}{\epsilon_0} = 0$ except that the difference has homogeneous boundary conditions. A general result for harmonic functions⁴ is that there is no local minimum or maximum in the volume inside the bounding surface. Thus with homogeneous boundary conditions the only solution to the homogeneous Poisson equation is the trivial function, $\psi(\vec{x}) = 0$ and the Green's theorem construction for the Dirichlet problem is unique.

There is a similar construction for the Neumann problem but with added complications due to the boundary conditions,

$$\begin{aligned} \phi(\vec{x}) &= \int_V d^n \vec{x}' G_N(\vec{x}|\vec{x}') \frac{\rho(\vec{x}')}{\epsilon_0} \\ &+ \int_{S \supset V} d^{n-1} \vec{x}' G_N(\vec{x}|\vec{x}') \frac{\partial\phi}{\partial\vec{x}'}(\vec{x}') \cdot \hat{n}|_S + \langle\phi\rangle|_S, \quad (\text{C.13}) \end{aligned}$$

with $G_N(\vec{x}|\vec{x}')$ where $\langle\phi\rangle|_S \equiv \frac{\int_S d^{n-1} \vec{x} \phi(\vec{x})|_S}{\int_S d^{n-1} \vec{x}}$. This extra term for the Neumann case is to fix the non-uniqueness of the solution. In the Neumann case, as above, if there were two solutions, the difference between the solutions $\psi(\vec{x}) = \phi_1(\vec{x}) - \phi_2(\vec{x})$ would be a solution of the homogeneous Poisson problem, $\vec{\nabla}^2 \psi(\vec{x}) = 0$, with homogeneous Neumann boundary conditions, $\frac{\partial\psi}{\partial\vec{x}} \cdot \hat{n}|_S = 0$. This equation has the solution $\psi(\vec{x}) = \text{constant}$. In addition for the Neumann case, Poisson's equation requires

$$\int_V d^n \vec{x} \frac{\rho(\vec{x})}{\epsilon_0} = - \int_V d^n \vec{x} \vec{\nabla}^2 \phi(\vec{x}) = - \int_{S \supset V} d^{n-1} \vec{x} \frac{\partial\phi}{\partial\vec{x}} \cdot \hat{n}|_S \quad (\text{C.14})$$

⁴Harmonic functions are functions that satisfy the homogeneous Poisson equation, $\vec{\nabla}^2 \phi(\vec{x}) = 0$.

by Gauss's theorem. This constraint must be satisfied for any well posed Neumann problem.

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C.3 Wave Equation

For a scalar field, $\psi(\vec{x}, t)$, the wave equation is

$$\square^2 \psi(\vec{x}, t) \equiv \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \psi(\vec{x}, t) = \rho(\vec{x}, t) \quad (\text{C.15})$$

We seek solutions of this problem for the field within some volume of space for all times. This is a hyperbolic differential equation and the solution is propagated from an initial time surface, see Section 2.2.2. The initial conditions are $\psi(\vec{x}, t_0)$ and $\frac{\partial \psi}{\partial t}(\vec{x}, t_0)$ throughout the volume of interest. In addition, there is an elliptic differential system included in the spatial part. This then requires the solution machinery of Section C.1 and will require the information from a surface with the boundary conditions either Dirichlet or Neumann or mixed on the closed surface bounding our volume for all $t \geq t_0$. The source term, $\rho(\vec{x}, t)$, given everywhere for all t . A special important case exists in the problem of unbounded systems. These will be discussed separately.

Instead of developing a Green's function solution directly, we construct a related object, the propagator. The propagator satisfies

$$\begin{aligned} \square^2 K(\vec{x}, t | \vec{x}', t') &= 0 \\ \lim_{t \rightarrow t'+} K(\vec{x}, t | \vec{x}', t') &= 0 \\ \lim_{t \rightarrow t'+} \frac{\partial K}{\partial t}(\vec{x}, t | \vec{x}', t') &= c^2 \delta^3(\vec{x} - \vec{x}') \end{aligned} \quad (\text{C.16})$$

with homogeneous boundary conditions of the Dirichlet type, $K(\vec{x}, t | \vec{x}', t') = 0$, or Neumann type, $\hat{n} \cdot \frac{\partial K}{\partial \vec{x}}(\vec{x}, t | \vec{x}', t') = 0$, for \vec{x} on the surface bounding the volume.

If the volume is stationary, Equations C.16 are time translation invariant and thus the propagator is a function of $t - t'$ only. The propagator satisfies the symmetries

$$\begin{aligned} K(\vec{x}, t | \vec{x}', t') &= K(\vec{x}, \vec{x}' : \tau \equiv t - t') \\ K(\vec{x}, \vec{x}' : \tau \equiv t - t') &= K(\vec{x}', \vec{x} : \tau \equiv t - t'). \end{aligned} \quad (\text{C.17})$$

The Green's function satisfies

$$\square^2 G(\vec{x}, t | \vec{x}', t') = \delta(t - t') \delta^3(\vec{x} - \vec{x}') \quad (\text{C.18})$$

with homogeneous boundary conditions and $G(\vec{x}, t | \vec{x}', t') = 0$ for $t < t'$.

The propagator and the Green's function are related by

$$G(\vec{x}, t | \vec{x}', t') = \theta(t - t') K(\vec{x}, t | \vec{x}', t') \quad (\text{C.19})$$

The solution for the field is

$$\begin{aligned} \psi(\vec{x}, t) = & \int_{t_0}^{t^+} dt' \int_V d^3 \vec{x}' G(\vec{x}, t | \vec{x}', t') \rho(\vec{x}', t') \\ & + \int_{t_0}^{t^+} dt' \int_{S \supset V} d^2 \vec{x}' G(\vec{x}, t | \vec{x}', t') \frac{\partial \psi}{\partial x'_n}(\vec{x}', t') \\ & - \int_{t_0}^{t^+} dt' \int_{S \supset V} d^2 \vec{x}' \frac{\partial G}{\partial x'_n}(\vec{x}, t | \vec{x}', t') \psi(\vec{x}', t') \\ & + \int_V d^3 \vec{x}' \frac{1}{c^2} G(\vec{x}, t | \vec{x}', t_0) \frac{\partial \psi}{\partial t}(\vec{x}', t_0) \\ & + \frac{\partial}{\partial t} \int_V d^3 \vec{x}' G(\vec{x}, t | \vec{x}', t_0) \psi(\vec{x}', t_0) \end{aligned} \quad (\text{C.20})$$

There is an eigenfunction expansion for the propagator with $-\nabla^2 \phi_i(\vec{x}) = \lambda_i \phi_i(\vec{x})$ plus homogeneous boundary conditions.

$$K(\vec{x}, t | \vec{x}', t') = c^2 \left\{ \frac{1}{V} (t - t') + \sum'_i \phi_i(\vec{x}) \phi_i^*(\vec{x}') \frac{\sin [c\sqrt{\lambda_i} (t - t')]}{c\sqrt{\lambda_i}} \right\} \quad (\text{C.21})$$

where the first term is present only if there is a $\lambda_0 = 0$ eigenvalue and the \sum' indicates that the sum omits the zero eigenvalue.

In an unbounded space, the propagator in n dimensions is

$$K_0^{(n)}(\vec{R} \equiv \vec{x} - \vec{x}', \tau \equiv (t - t')) = c(2\pi)^{-n} \int d^n k \frac{\sin(kcR)}{k} e^{i\vec{k} \cdot \vec{R}} \quad (\text{C.22})$$

The Green's functions are

$$G_0^{(3)}(\vec{R} \equiv \vec{x} - \vec{x}', \tau \equiv (t - t')) = \theta(\tau) \frac{\delta(\tau - \frac{R}{c})}{4\pi R} \quad (\text{C.23})$$

$$G_0^{(2)}(\vec{R}, \tau) = \theta(\tau) \frac{1}{2\pi} \frac{\theta(\tau - \frac{R}{c})}{\left[\tau^2 - \frac{R^2}{c^2}\right]^{\frac{1}{2}}} \quad (\text{C.24})$$

$$G_0^{(1)}(x - x', \tau) = \theta(\tau) \frac{c}{2} \theta\left(\tau - \frac{|x - x'|}{c}\right) \quad (\text{C.25})$$