Appendix E

Vector Spherical Harmonics

E.1 Spherical Harmonics

E.1.1 Legendre Polynomials

The Legendre polynomials are solutions to Legendre's differential equation

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{d}{dx}P_l\left(x\right)\right]+\lambda P_l\left(x\right)=0. \tag{E.1}$$

Equation E.1 has singular points at $x=\pm 1$ and can be solved for the interval $-1\leq x\leq 1$ with a power series solution that terminates. This requires that $\lambda = l\left(l+1\right)$ where l is an integer. The standard forms fit $P_l(1)=1$ and the solutions are odd or even polynomials of degree l and are

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \frac{1}{2}\left(3x^2-1\right)\\
P_3(x) &= \frac{1}{2}\left(5x^3-3x\right)\\
P_4(x) &= \frac{1}{8}\left(35x^4-30x^2+3\right)\\
\vdots\quad &\quad \vdots\\
P_l(x) &= \frac{(2l-1)!!}{l!}\left[x^l-\frac{l\left(l-1\right)}{2\left(2l-1\right)}x^{l-2}\right.\\
&\qquad\left.+\frac{l\left(l-1\right)\left(l-2\right)\left(l-3\right)}{2\cdot 4\cdot\left(2l-1\right)\left(2l-3\right)}x^{l-4}-\cdots\right]
\end{aligned} \tag{E.2}$$

There is also a closed form solution, Rodrigues' Formula,

$$P_l\left(x\right) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l \left(x^2-1\right)^l . \tag{E.3}$$

The Legendre polynomials have the parity

$$P_l\left(-x\right) = (-1)^l P_l\left(x\right), \tag{E.4}$$

and are complete on the interval $-1 \leq x \leq 1$,

$$\int_{-1}^{1} dx P_l(x) P_l'(x) = \frac{\delta_{ll'}}{2l+1} \tag{E.5}$$

$$\sum_l \left(l+\frac{1}{2}\right) P_l(x) P_l(x') = \delta(x-x') \tag{E.6}$$

Many applications of the Legendre polynomials deal with the unit circle or the unit sphere and in those cases the argument x is the cosine of the angle to the axis. In fact, given two vectors $\vec{r}$ and $\vec{r'}$ with the angle between them θ, the form

$$\frac{1}{|\vec{r}-\vec{r'}|} = \left(r^2 - 2rr'\cos\theta + r'^2\right)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l\left(\cos\theta\right) \tag{E.7}$$

is the generating function for the P_l functions and where $r_<$ is the lesser of (r, r') and $r_>$ is the greater of (r, r').

With $\cos\theta$ as argument Rodrigues' formula, Equation E.3, becomes

$$P_l\left(\cos\theta\right) = \frac{(-1)^l}{2^l l!}\left(\frac{d}{d(\cos\theta)}\right)^l (\sin\theta)^{2l} \tag{E.8}$$

The Legendre polynomials satisfy the recurrence relations

$$(2l+1)\cos\theta P_l(\cos\theta) = (l+1)\, P_{l+1}(\cos\theta) + l P_{l-1}(\cos\theta) \tag{E.9}$$

$$\frac{d}{d\cos\theta}\left(P_{l+1}(\cos\theta)\right) = (l+1)\, P_l(\cos\theta) + \cos\theta \frac{d}{d\cos\theta}\left(P_l(\cos\theta)\right) \tag{E.10}$$

$$\frac{d}{d\cos\theta}\left(P_{l-1}(\cos\theta)\right) = -l P_l(\cos\theta) + \cos\theta \frac{d}{d\cos\theta}\left(P_l(\cos\theta)\right) \tag{E.11}$$

$$\begin{aligned}
\frac{\sin^2\theta}{l}\frac{d}{d\cos\theta}\left(P_l(\cos\theta)\right) &= P_{l-1}(\cos\theta) - \cos\theta P_l(\cos\theta) \qquad \text{(E.12)} \\
\frac{\sin^2\theta}{(l+1)}\frac{d}{d\cos\theta}\left(P_l(\cos\theta)\right) &= -P_{l+1}(\cos\theta) + \cos\theta P_l(\cos\theta) \qquad \text{(E.13)} \\
\int_0^{\cos\theta}(2l+1)\,P_l(\cos\theta')d\left(\cos\theta'\right) &= \left[P_{l+1}(\cos\theta) - P_{l-1}(\cos\theta)\right]. \qquad \text{(E.14)} \\
(2l+1)\,P_l(\cos\theta) &= \frac{d}{d\cos\theta}\left\{P_{l+1}(\cos\theta) - P_{l-1}(\cos\theta)\right\} \qquad \text{(E.15)}
\end{aligned}$$

Equations E.14 and E.15,which are the same recursion relation, and are valid only for $l > 0$. For $l = 0$ it is trivial, replace P_{-1} by 1.

There are also some useful integral relationships,

$$\begin{aligned}
\int_0^{\pi} P_{2l}\left(\cos\theta\right)d\theta &= \pi\left\{\frac{(2l)!}{\left[2^l l!\right]^2}\right\}^2 \qquad \text{(E.16)} \\
\int_0^{\pi} P_{2l+1}\left(\cos\theta\right)\cos\theta d\theta &= \pi\frac{2l!(2l+2)!}{\left[2^{2l+1}l!\left(2l+1\right)!\right]^2} \qquad \text{(E.17)} \\
\int_0^{\pi} P_l\left(\cos\theta\right)\sin\left(m\theta\right)d\theta & \\
&= 2\frac{\left(m+l-1\right)\left(m+l-3\right)\cdots\left(m-l+1\right)}{\left(m+l\right)\left(m+l-2\right)\cdots\left(m-l\right)}. \qquad \text{(E.18)}
\end{aligned}$$

Equation E.18 is valid only when $l < m$ and $(l+m)$ is odd; otherwise it is zero.

Other useful formulas are

$$\begin{aligned}
(2l+1)\left(\cos\theta\right)^{2l} &= P_0(\cos\theta) + \frac{5\cdot 2l}{2l+3}P_2(\cos\theta) \\
&\quad + \frac{9\cdot 2l\left(2l-2\right)}{\left(2l+3\right)\left(2l+5\right)}P_4(\cos\theta) + \cdots \qquad \text{(E.19)} \\
(2l+3)\left(\cos\theta\right)^{2l+1} &= 3P_1(\cos\theta) + \frac{7\cdot 2l}{2l+5}P_3(\cos\theta) \\
&\quad + \frac{11\cdot 2l\left(2l-2\right)}{\left(2l+5\right)\left(2l+7\right)}P_5(\cos\theta) + \cdots \qquad \text{(E.20)} \\
\frac{d}{d\cos\theta}P_l(\cos\theta) &= (2l-1)\,P_{l-1}(\cos\theta) + (2l-5)\,P_{l-3}\left(\cos\theta\right) \\
&\quad + (2l-9)\,P_{l-5}\left(\cos\theta\right) + \cdots \qquad \text{(E.21)}
\end{aligned}$$

with Equation E.21 terminating when at the first negative index.

Since the $P_l(\cos\theta)$'s are complete on the unit circle their relationship to the rotations in one dimension is important. The generator of rotations in this basis is $i\frac{d}{d\theta}$ and, using Equation E.21, we have

$$\begin{aligned} -i\frac{d}{d\theta}P_l(\cos\theta) &= i\sin\theta\left\{(2l-1)\,P_{l-1}(\cos\theta)+(2l-5)\,P_{l-3}\,(\cos\theta)\right.\\ &\quad\left.+\,(2l-9)\,P_{l-5}\,(\cos\theta)+\cdots\right\} \end{aligned} \tag{E.22}$$

An even more important form can be found using the associated Legendre polynomial, see Section E.1.2,

$$\frac{d}{d\theta}P_l(\cos\theta)=P_l^1(\cos\theta). \tag{E.23}$$

An important identity for two points, one at (θ_1,ϕ_1) and the other at (θ_2,ϕ_2), on the unit sphere is

E.1.2 Associated Legendre Polynomials

The associated Legendre functions of the first kind are defined as

$$P_l^m(x)\equiv\frac{\left(1-x^2\right)^{\frac{1}{2}m}}{2^l l!}\frac{d^{l+m}}{dx^{l+m}}\left(x^2-1\right)^l \quad m,n=0,1,2,3\ldots \tag{E.24}$$

Again for convenience, using as argument $x=\cos\theta$ and in terms of the Legendre polynomials by

$$P_l^m(\cos\theta)\equiv(\sin\theta)^m\left(\frac{d}{d\ \cos\theta}\right)^m P_l(\cos\theta)\quad 0\le m\le l. \tag{E.25}$$

They can also be written

$$P_l^m\,(\cos\theta)=\frac{(-1)^{l+m}}{2^l l!}\frac{(l+m)!}{(l-m)!}\,(\sin\theta)^{-m}\left(\frac{d}{d\cos\theta}\right)^{l-m}(\sin\theta)^{2l} \tag{E.26}$$

For negative m, the P_l^m are given by

$$P_l^{-m}\equiv(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m \tag{E.27}$$

Their parity is $(l+m)$

$$P_l^m(-x)=(-1)^{l+m}P_l^m(x) \tag{E.28}$$

They are solutions to Legendre's differential equation,

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{d}{dx}P_l^m(x)\right)+\left(l(l+1)-\frac{m^2}{1-x^2}\right)P_l^m(x)=0 \tag{E.29}$$

which in terms of $\cos\theta$ is

$$\frac{d}{d\cos\theta}\left(\sin^2\theta\frac{d}{d\cos\theta}P_l^m(\cos\theta)\right)+\left(l(l+1)-\frac{m^2}{\sin^2\theta}\right)P_l^m(\cos\theta)=0 \tag{E.30}$$

Some assocated Legendre functions for small l and m are

$$\begin{aligned}
P_0^0 &= 1\\
P_1^0 &= x=\cos\theta;\quad P_1^1=\sqrt{1-x^2}=\sin\theta\\
P_2^0 &= \frac{1}{2}\left(3x^2-1\right)=\frac{1}{4}\left(3\cos 2\theta+1\right);\quad P_2^1=3x\sqrt{1-x^2}=\frac{3}{2}\sin 2\theta;\\
&\qquad P_2^2=3\left(1-x^2\right)=\frac{3}{2}\left(1-\cos 2\theta\right)\\
P_3^0 &= \frac{1}{2}\left(5x^3-3x\right)=\frac{1}{8}\left(5\cos 3\theta+3\cos\theta\right);\\
&\qquad P_3^1=\frac{3}{2}\sqrt{1-x^2}\left(5x^2-1\right)=\frac{3}{8}\left(\sin\theta+5\cos 3\theta\right)\\
&\qquad P_3^2=15x\left(1-x^2\right)=\frac{15}{4}\left(\cos\theta-\cos 3\theta\right)\\
&\qquad P_3^3=15x\left(1-x^2\right)^{\frac{3}{2}}=\frac{15}{4}\left(3\sin\theta-\sin 3\theta\right).
\end{aligned} \tag{E.31}$$

Important recursion relations are

$$(2l+1)\sqrt{1-x^2}P_l^m(x)=P_{l+1}^{m+1}(x)-P_{l-1}^{m+1}(x), \tag{E.32}$$

$$=(l+m)(l+m-1)P_{l-1}^{m+1}(x)-(l-m+1)(l-m+2)P_{l+1}^{m-1}(x), \tag{E.33}$$

and

$$(2l+1)xP_l^m(x) = (l-m+1)P_{l+1}^m(x)+(l+m)P_{l-1}^m(x) \tag{E.34}$$

$$(1-x^2)\frac{d}{dx}P_l^m(x) = (l+1)xP_l^m(x)-(l-m+1)P_{l+1}^m(x) \tag{E.35}$$

Other useful differential and integral relations are

$$\frac{d}{dx}\left[\left(1-x^2\right)^{\frac{1}{2}m}P_l^m(x)\right] = -(l-m+1)(l+m)\left(1-x^2\right)^{\frac{1}{2}(m-1)}P_l^{m-1}(x) \tag{E.36}$$

$$\int_{-1}^{1} P_n^m(x) P_l^m(x) dx = \frac{2(l+m)!}{(2l+1)(l-m)!}\delta_{nl} \tag{E.37}$$

$$\int_{-1}^{1} \frac{P_n^m(x) P_l^m(x)}{1-x^2} dx = \frac{(l+m)!}{m(l-m)!}\delta_{nl} \tag{E.38}$$

There is an integral form corresponding to Equation E.25,

$$P_l^m(x) = \frac{(l+m)!}{(l-m)!}\left(1-x^2\right)\int_x^1 dz_1 \int_{z_1}^1 dz_2 \cdots \int_{z_{m-1}}^1 dz_m P_l(z_m). \tag{E.39}$$

There is a generating function similar to Equation E.7,

$$\frac{2^m m! \sin^m\theta}{\left[r^2 - 2rr'\cos\theta + r'^2\right]^{m+\frac{1}{2}}} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_{l+m}^m(\cos\theta) \tag{E.40}$$

where again $r_<$ is the lesser of (r, r') and $r_>$ is the greater of (r, r').

E.1.3 Spherical Harmonics

The spherical harmonics are defined by

$$Y_l^m(\hat{r}) \equiv (-1)^m i^l \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{\frac{1}{2}} e^{im\phi} P_l^m(\cos\theta) \tag{E.41}$$

with θ and ϕ being the usual polar angles of the unit vector $\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\cos\phi\hat{y} + \cos\theta\hat{z}$.

With the conventions here, Equation E.27, the complex conjugation of the spherical harmonics are

$$Y_l^{m^*} = (-1)^{l+m} Y_l^{-m}. \tag{E.42}$$

The parity is

$$Y_l^m(-\hat{r}) = (-1)^l Y_l^m(\hat{r}). \tag{E.43}$$

The spherical harmonics form a complete orthonormal set on the unit sphere,

$$\begin{aligned}\int d^2\Omega_{\hat{r}} Y_l^{m^*}(\hat{r}) Y_{l'}^{m'}(\hat{r}) &= \delta_{ll'}\delta_{mm'} \\ \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^m(\hat{r}) Y_l^{m^*}(\hat{r}') &= \delta^2\left(\Omega_{\hat{r}} - \Omega_{\hat{r}'}\right)\end{aligned} \tag{E.44}$$

where $\delta^2\left(\Omega_{\hat{r}}-\Omega_{\hat{r}'}\right)$ is the solid angle delta function,

$$\int d^2\Omega_{\hat{r}}\delta^2\left(\Omega_{\hat{r}}-\Omega_{\hat{r}'}\right)f\left(\hat{r}\right)=f\left(\hat{r}'\right). \tag{E.45}$$

and $d^2\Omega_{\hat{r}}=\sin\theta d\theta d\phi$.

The spherical harmonics solve the partial differential equation

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_l^m(\hat{r})=l\left(l+1\right)Y_l^m(\hat{r}). \tag{E.46}$$

The $Y_l^m(\hat{r})$ are an irreducible representation of rotations on the unit sphere in the sense that the generators of the rotations are

$$\vec{L}_i\equiv\left[\vec{r}\times\left(-i\vec{\nabla}\right)\right]_i. \tag{E.47}$$

In particular, the z component is

$$\begin{aligned}\left[\vec{r}\times\left(-i\vec{\nabla}\right)\right]_z &= -i\hat{z}\cdot\left[\vec{r}\times\left(\hat{r}\frac{\partial}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right]\\ &= -i\hat{z}\cdot\left[\hat{\phi}\frac{\partial}{\partial\theta}-\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]\\ &= -i\frac{\partial}{\partial\phi}\end{aligned} \tag{E.48}$$

and thus

$$\vec{L}_zY_l^m=\left[\vec{r}\times\left(-i\vec{\nabla}\right)\right]_zY_l^m=-i\frac{\partial}{\partial\phi}Y_l^m=mY_l^m. \tag{E.49}$$

The forms for the other components are

$$\begin{aligned}\left[\vec{r}\times\left(-i\vec{\nabla}\right)\right]_x &= -i\hat{x}\cdot\left[\hat{\phi}\frac{\partial}{\partial\theta}-\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]\\ &= i\left[\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right]\end{aligned} \tag{E.50}$$

$$\begin{aligned}\left[\vec{r}\times\left(-i\vec{\nabla}\right)\right]_y &= -i\hat{y}\cdot\left[\hat{\phi}\frac{\partial}{\partial\theta}-\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]\\ &= -i\left[\cos\theta\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right]\end{aligned} \tag{E.51}$$

Equation E.46 is a consequence of the fact that

$$\begin{aligned}\vec{L}^2 Y_l^m(\hat{r}) &\equiv \sum_{i=x,y,z}\left[\vec{r}\times\left(-i\vec{\nabla}\right)\right]_i\left[\vec{r}\times\left(-i\vec{\nabla}\right)\right]_i Y_l^m(\hat{r})\\ &= -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_l^m(\hat{r})\\ &= l\left(l+1\right)Y_l^m(\hat{r}), \qquad \text{(E.52)}\end{aligned}$$

see Section **??**.

Since the Y_l^m's are orthogonal and complete, the most general rotationally invariant function of $\hat{r}_1$ and $\hat{r}_2$ can be expanded in the form

$$f\left(\hat{r}_1,\hat{r}_2\right)=\sum_l c_l Y_l^m\left(\hat{r}_1\right)Y_l^{m^*}\left(\hat{r}_2\right). \qquad \text{(E.53)}$$

Four especially important examples are

$$4\pi\sum_{m=-l}^{l}Y_l^m\left(\hat{r}_1\right)Y_l^{m^*}\left(\hat{r}_2\right)=\left(2l+1\right)P_l\left(\hat{r}_1\cdot\hat{r}_2\right), \qquad \text{(E.54)}$$

$$e^{i\vec{k}\cdot\vec{r}}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}i^l j_l(kr)Y_l^m(\hat{k})Y_l^{m^*}\left(\hat{r}\right), \qquad \text{(E.55)}$$

where $j_l\left(kr\right)=\left[\frac{\pi}{2z}\right]^{\frac{1}{2}}J_{l+\frac{1}{2}}\left(z\right)$ and $J_{l+\frac{1}{2}}$ is the Bessel function of half integer order,

$$\frac{1}{\left|\vec{r}-\vec{r}'\right|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_l^m\left(\hat{r}\right)Y_l^{m^*}\left(\hat{r}'\right), \qquad \text{(E.56)}$$

and

$$\frac{e^{i\vec{k}\cdot\left(\vec{r}-\vec{r}'\right)}}{4\pi\left|\vec{r}-\vec{r}'\right|}=ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l}j_l\left(kr_<\right)h_l\left(kr_>\right)Y_l^m\left(\hat{r}\right)Y_l^{m^*}\left(\hat{r}'\right) \qquad \text{(E.57)}$$

where $h_l\left(kr\right)$ is the outgoing Hankel function.

E.2 Intrinsic Spherical Harmonics

Not all fields are scalar fields. In particular, the electromagnetic field is a vector field. For each of the electromagnetic fields there are three fields at each point that that form a vector and are thus related by rotations. Instead of talking about the three component fields, it is possible to separate the effects of rotation on the component fields and the effects on the basis vectors. The first step in this process is to bring the usual rotation relationships for the basis vectors into a language that is similar to that the irreducible representations of the rotation group. The basis vectors in the a three dimensional space already form an irreducible representation of the rotation group but they do not follow the pattern of the Y_l^m's under rotations, see Section **??**. By transforming the basis to more appropriate basis, we can find a set which acts like the Y_1^m triplet. Moving them into this form will allow us to use powerful techniques of group theory to combine the effects the rotation on the vector parts of the field, called intrinsic, with the effects of the functional parts to be able to identify the forms that are irreducible under rotations of both parts of the field.

Defining a new basis set, $\hat{\chi}_\mu$,

$$\begin{aligned} \hat{\chi}_1 &\equiv -\frac{1}{\sqrt{2}}\left(\hat{x} + i\hat{y}\right) \\ \hat{\chi}_0 &\equiv \hat{z} \\ \hat{\chi}_{-1} &\equiv \frac{1}{\sqrt{2}}\left(\hat{x} - i\hat{y}\right) \end{aligned} \tag{E.58}$$

and its inverse

$$\begin{aligned} \hat{x} &= -\frac{1}{\sqrt{2}}\left(\hat{\chi}_1 - \hat{\chi}_{-1}\right) \\ \hat{y} &= \frac{i}{\sqrt{2}}\left(\hat{\chi}_1 + \hat{\chi}_{-1}\right) \\ \hat{z} &= \hat{\chi}_0 \end{aligned} \tag{E.59}$$

The phases of the $\hat{\chi}_\mu$ are

$$\hat{\chi}_\mu^* = (-1)^\mu \, \hat{\chi}_{-\mu} \tag{E.60}$$

This set, $\hat{\chi}_\mu$, is an orthonormal basis for the expansion of any vector in the sense that

$$\hat{\chi}_{\mu'} \cdot \hat{\chi}_\mu^* = \hat{\chi}_{\mu'}^* \cdot \hat{\chi}_\mu = \delta_{\mu'\mu}. \tag{E.61}$$

Any vector, $\vec{A} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$, can be expanded

$$\vec{A} = \sum_\mu a_\mu \hat{\chi}_\mu^* = \sum_\mu a_\mu^* \hat{\chi}_\mu \tag{E.62}$$

with

$$\begin{aligned} a_\mu &\equiv \hat{\chi}_\mu \cdot \vec{A} \\ &= \sum_{\mu'} a_{\mu'} \hat{\chi}_\mu \cdot \hat{\chi}^*_{\mu'} \\ &= \sum_{\mu'} a_{\mu'} \delta_{\mu\mu'} . \end{aligned} \tag{E.63}$$

The coefficients a_μ are

$$\begin{aligned} a_1 &= \hat{\chi}_1 \cdot \vec{A} = -\frac{1}{\sqrt{2}} \left(a_x + i a_y\right) \\ a_0 &= \hat{\chi}_0 \cdot \vec{A} = a_z \\ a_{-1} &= \hat{\chi}_{-1} \cdot \vec{A} = \frac{1}{\sqrt{2}} \left(a_x - i a_y\right) \end{aligned} \tag{E.64}$$

with the inverse

$$\begin{aligned} a_x &= \frac{1}{\sqrt{2}} \left(a_1 - a_{-1}\right) \\ a_z &= a_0 \\ a_y &= \frac{i}{\sqrt{2}} \left(a_x + a_{-1}\right) . \end{aligned} \tag{E.65}$$

The Inner product of two vectors is

$$\begin{aligned} \vec{A} \cdot \vec{B} &= a_x b_x + a_y b_y + a_z b_z \\ &= \sum_{\mu=1,0-,1} a_\mu b^*_\mu = \sum_{\mu=1,0-,1} a^*_\mu b_\mu \end{aligned} \tag{E.66}$$

Completeness of the basis is expressed by the unit dyadic

$$\overset{\leftrightarrow}{\mathbb{I}} \equiv \sum_{i,j=x,y,z} \delta_{ij} \hat{i}\hat{j} = \sum_{\mu.\mu'=1,0,-1} \delta_{\mu\mu'} \hat{\chi}_\mu \hat{\chi}^*_{\mu'} \tag{E.67}$$

Any point on the unit sphere is composed linearly from the basis $\hat{x}$, $\hat{y}$, $\hat{z}$ and this provides a three dimensional irreducible representation of rotations on the unit sphere. In this basis the rotations are generated by

$$\left(\vec{S}_i\right)_{jk} = i\epsilon_{ijk} \tag{E.68}$$

where i, j, k range through the values x, y, z. In the space of basis vectors, the spin operator is actually a dyadic; it maps the basis vectors onto basis vectors. Its form is thus

$$\overleftrightarrow{S} = i\, \overleftrightarrow{\mathbb{I}} \times \tag{E.69}$$

or taking the components of this equation

$$\begin{aligned}\overleftrightarrow{S}_x &= \hat{x} \cdot \overleftrightarrow{S} = i\hat{x} \times \\ \overleftrightarrow{S}_y &= \hat{y} \cdot \overleftrightarrow{S} = i\hat{y} \times \\ \overleftrightarrow{S}_z &= \hat{z} \cdot \overleftrightarrow{S} = i\hat{z} \times .\end{aligned} \tag{E.70}$$

For example, this produces the following terms for the z axis rotation:

$$\begin{aligned}\left(\overleftrightarrow{S}_z\right)\hat{x} &= i\hat{y} \\ \left(\overleftrightarrow{S}_z\right)\hat{y} &= -i\hat{x} \\ \left(\overleftrightarrow{S}_z\right)\hat{z} &= 0\end{aligned} \tag{E.71}$$

The $\hat{\chi}_\mu$ are defined in such a way that they behave in the same way as the Y_1^1 under rotations on the unit sphere and are thus another three dimensional irreducible representations of the rotations on the unit sphere, see Section **??**. Thus, they also satisfy the generator algebra

$$\begin{aligned}\overleftrightarrow{S}_x\, \hat{\chi}_{-1} &= \frac{1}{\sqrt{2}}\hat{\chi}_0 \\ \overleftrightarrow{S}_x\, \hat{\chi}_0 &= \frac{1}{\sqrt{2}}\left(\hat{\chi}_1 + \hat{\chi}_{-1}\right) \\ \overleftrightarrow{S}_x\, \hat{\chi}_{-1} &= \frac{1}{\sqrt{2}}\hat{\chi}_0 \\ \overleftrightarrow{S}_y\, \hat{\chi}_1 &= \frac{i}{\sqrt{2}}\hat{\chi}_0 \\ \overleftrightarrow{S}_y\, \hat{\chi}_0 &= \frac{i}{\sqrt{2}}\left(\hat{\chi}_1 - \hat{\chi}_{-1}\right) \\ \overleftrightarrow{S}_y\, \hat{\chi}_{-1} &= \frac{-i}{\sqrt{2}}\hat{\chi}_0 \\ \overleftrightarrow{S}_z\, \hat{\chi}_1 &= \hat{\chi}_1 \\ \overleftrightarrow{S}_z\, \hat{\chi}_0 &= 0 \\ \overleftrightarrow{S}_z\, \hat{\chi}_{-1} &= -\hat{\chi}_{-1}\end{aligned} \tag{E.72}$$

using Equation E.61, the matrix operators are

$$\begin{aligned}
\left(\vec{S}_x\right)_{\mu\mu'} &\equiv \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \\
\left(\vec{S}_y\right)_{\mu\mu'} &\equiv \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ 0 & \frac{-i}{\sqrt{2}} & 0 \end{pmatrix} \\
\left(\vec{S}_z\right)_{\mu\mu'} &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\end{aligned} \tag{E.73}$$

and

$$\begin{aligned}
\left(\vec{S}^2\right)_{\mu\mu'} &\equiv \sum_{\substack{i=1,2,3 \\ \mu''=1,0,-1}} \left(\vec{S}_i\right)_{\mu\mu''}\left(\vec{S}_i\right)_{\mu''\mu'} \\
&= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\end{aligned} \tag{E.74}$$

As expected, the generators of rotations for the new basis elements have two diagonal elements

$$\begin{aligned}
\sum_{\mu'=1,0,-1} \left(\vec{S}_z\right)_{\mu\mu'} \hat{\chi}_{\mu'} &= \mu\hat{\chi}_\mu \\
\sum_{\mu'=1,0,-1} \left(\vec{S}^2\right)_{\mu\mu'} \hat{\chi}_{\mu'} &= 2\hat{\chi}_\mu
\end{aligned} \tag{E.75}$$

E.3 Vector Spherical Harmonics

A vector field is a set of functions of coordinates attached to vector. Rotations of the system are generated by

$$\left(\vec{J}\right)_{\mu\mu'} = \vec{L}\delta_{\mu\mu'} + 1\left(\vec{S}\right)_{\mu\mu'} \tag{E.76}$$

where Equation E.47 defines $\vec{L}$ and the $\left(\vec{S}\right)_{\mu\mu'}$ is given by Equation E.73. The 1 preceding the intrinsic operators is the signal that the coordinates are unchanged similar to the role of the $\delta_{\mu\mu'}$ multiplying the $\vec{L}$.

By using the basis $\hat{\chi}_\mu$ for the vector the effect of the rotations on the intrinsic degrees of freedom, the vector directions, and the rotations of the field coordinates as represented by the Y_l^m can be combined. In this case, the vector degrees of freedom are called spin degrees of freedom.

We can now construct the direct product forms.

$$\vec{Y}_{l,m;1,\mu}(\hat{r}) \equiv Y_l^m(\hat{r})\,\hat{\chi}_\mu \tag{E.77}$$

These forms have the following eigenvalue relations with the generators:

$$\begin{aligned}
L_z\vec{Y}_{l,m;1,\mu}(\hat{r}) &= m\vec{Y}_{l,m;1,\mu}(\hat{r})\\
L^2\vec{Y}_{l,m;1,\mu}(\hat{r}) &= l(l+1)\,\vec{Y}_{l,m;1,\mu}(\hat{r})\\
S_z\vec{Y}_{l,m;1,\mu}(\hat{r}) &= \mu\vec{Y}_{l,m;1,\mu}(\hat{r})\\
S^2\vec{Y}_{l,m;1,\mu}(\hat{r}) &= 2\vec{Y}_{l,m;1,\mu}(\hat{r})\,.
\end{aligned} \tag{E.78}$$

These forms are reducible representations of the rotation group when the rotations are defined by the generators as given in Equation E.76. Defining

$$\begin{aligned}
\vec{Y}_{Jl}^M(\hat{r}) &\equiv \sum_{m\mu} c(l,1,J;m,\mu,M)\,\vec{Y}_{l,m;1,\mu}(\hat{r})\\
&= \sum_{\mu} c(l,1,J;M-\mu,\mu,M)\,\vec{Y}_{l,M-\mu;1,\mu}(\hat{r})
\end{aligned} \tag{E.79}$$

where the $c(l,1,J;m,\mu,M)$ are the well known Clebsch-Gordon coefficients. These reduced forms have the following eigenvalue properties

$$\begin{aligned}
(L_z+S_z)\,\vec{Y}_{Jl}^M(\hat{r}) &= M\vec{Y}_{Jl}^M(\hat{r})\\
\left(\vec{L}+\vec{S}\right)^2\vec{Y}_{Jl}^M(\hat{r}) &= J(J+1)\,\vec{Y}_{Jl}^M(\hat{r})\\
\vec{L}^2\vec{Y}_{Jl}^M(\hat{r}) &= l(l+1)\,\vec{Y}_{Jl}^M(\hat{r})\\
\vec{S}^2\vec{Y}_{Jl}^M(\hat{r}) &= 2\vec{Y}_{Jl}^M(\hat{r})
\end{aligned} \tag{E.80}$$

and thus are an irreducible representation of rotations.

These forms can be inverted to yield the product states,

$$\begin{aligned}
\vec{Y}_{l,m;1,\mu}(\hat{r}) &= \sum_{JM} <l,m,1,\mu|J,M,l,1>\vec{Y}_{Jl}^M(\hat{r})\\
&= \sum_{J} <l,m,1,\mu|J,m+\mu,l,1>\vec{Y}_{Jl}^M(\hat{r})\\
&= <l,m,1,\mu|l+1,m+\mu,l,1>\vec{Y}_{l+1l}^M(\hat{r})\\
&\quad + <l,m,1,\mu|l,m+\mu,l,1>\vec{Y}_{ll}^M(\hat{r})\\
&\qquad + <l,m,1,\mu|l-1,m+\mu,l,1>\vec{Y}_{l-1l}^M(\hat{r})\,.
\end{aligned} \tag{E.81}$$

These irreducible representations are orthonormal

$$\int d^2\Omega\hat{r}\vec{Y}_{Jl}^{M^*}\left(\hat{r}\right)\cdot\vec{Y}_{J'l'}^{M'}\left(\hat{r}\right)=\delta_{JJ'}\delta_{MM'}\delta_{ll'} \tag{E.82}$$

and complete on the unit sphere,

$$\sum_{JlM}\vec{Y}_{Jl}^{M}\left(\hat{r}\right)\vec{Y}_{Jl}^{M^*}\left(\hat{r}'\right)=\overleftrightarrow{I}\,\delta^2\left(\Omega_{\hat{r}}-\Omega_{\hat{r}'}\right). \tag{E.83}$$

In addition, the $\vec{Y}_{Jl}^{M}\left(\hat{r}\right)$ are an irreducible representation of the group of rotations on the unit sphere, see Appendix D.2.

Two useful relations are

$$\hat{r}Y_l^m\left(\hat{r}\right)=-\left[\frac{l+1}{2l+1}\right]^{\frac{1}{2}}\vec{Y}_{ll+1}^{m}\left(\hat{r}\right)+\left[\frac{l}{2l+1}\right]^{\frac{1}{2}}\vec{Y}_{ll-1}^{m}\left(\hat{r}\right), \tag{E.84}$$

$$\begin{aligned}\vec{\nabla}f\left(r\right)Y_l^m\left(\hat{r}\right) &= -\left[\frac{l+1}{2l+1}\right]^{\frac{1}{2}}\left(\frac{d}{dr}-\frac{l}{r}\right)f\left(r\right)\vec{Y}_{ll+1}^{m}\left(\hat{r}\right)\\ &\quad+\left[\frac{l}{2l+1}\right]^{\frac{1}{2}}\left(\frac{d}{dr}+\frac{l+1}{r}\right)f\left(r\right)\vec{Y}_{ll-1}^{m}\left(\hat{r}\right)\end{aligned} \tag{E.85}$$

Since

$$\left[\vec{\nabla}\times,\vec{J}\right]=0 \tag{E.86}$$

the curl operator connects only states with the same J and M;

$$\begin{aligned}\vec{\nabla}\times\left(f\left(r\right)\vec{Y}_{ll+1}^{m}\left(\hat{r}\right)\right) &= i\left[\frac{l}{2l+1}\right]^{\frac{1}{2}}\left(\frac{d}{dr}+\frac{l+2}{r}\right)\left(f\left(r\right)\vec{Y}_{ll}^{m}\right)\\ \vec{\nabla}\times\left(f\left(r\right)\vec{Y}_{ll}^{m}\left(\hat{r}\right)\right) &= i\left[\frac{l}{2l+1}\right]^{\frac{1}{2}}\left(\frac{d}{dr}-\frac{l}{r}\right)\left(f\left(r\right)\vec{Y}_{ll+1}^{m}\right)\\ &\quad+i\left[\frac{l+1}{2l+1}\right]^{\frac{1}{2}}\left(\frac{d}{dr}+\frac{l+1}{r}\right)\left(f\left(r\right)\vec{Y}_{ll-1}^{m}\right)\\ \vec{\nabla}\times\left(f\left(r\right)\vec{Y}_{ll-1}^{m}\left(\hat{r}\right)\right) &= i\left[\frac{l+1}{2l+1}\right]^{\frac{1}{2}}\left(\frac{d}{dr}-\frac{l-1}{r}\right)\left(f\left(r\right)\vec{Y}_{ll}^{m}\right)\end{aligned} \tag{E.87}$$

Similarly,

$$\begin{aligned}\vec{\nabla}\cdot\left(f\left(r\right)\vec{Y}_{ll+1}^{m}\left(\hat{r}\right)\right) &= -\left[\frac{l+1}{2l+1}\right]^{\frac{1}{2}}\left(\frac{d}{dr}+\frac{l+2}{r}\right)\left(f\left(r\right)Y_l^m\left(\hat{r}\right)\right)\\ \vec{\nabla}\cdot\left(f\left(r\right)\vec{Y}_{ll}^{m}\left(\hat{r}\right)\right) &= 0\\ \vec{\nabla}\cdot\left(f\left(r\right)\vec{Y}_{ll-1}^{m}\left(\hat{r}\right)\right) &= \left[\frac{l}{2l+1}\right]^{\frac{1}{2}}\left(\frac{d}{dr}-\frac{l-1}{r}\right)\left(f\left(r\right)Y_l^m\left(\hat{r}\right)\right)\end{aligned} \tag{E.88}$$

With this set of equations, we can identify a more useful set of expansion terms. The sets

$$\begin{aligned} \vec{Y}^{(e)}_{JM}(\hat{r}) &\equiv \left[\frac{J+1}{2J+1}\right]^{\frac{1}{2}} \vec{Y}^{M}_{JJ-1}(\hat{r}) - \left[\frac{J}{2J+1}\right]^{\frac{1}{2}} \vec{Y}^{M}_{JJ+1}(\hat{r}) \\ \vec{Y}^{(m)}_{JM}(\hat{r}) &\equiv \vec{Y}^{M}_{JJ}(\hat{r}) \\ \vec{Y}^{(o)}_{JM}(\hat{r}) &\equiv \left[\frac{J}{2J+1}\right]^{\frac{1}{2}} \vec{Y}^{M}_{JJ-1}(\hat{r}) + \left[\frac{J+1}{2J+1}\right]^{\frac{1}{2}} \vec{Y}^{M}_{JJ+1}(\hat{r}) \end{aligned} \tag{E.89}$$

are called the Hansen vector spherical harmonics. The first two of these are transverse, $\vec{\nabla} \cdot \left(f(r)\,\vec{Y}^{(e)}_{JM}(\hat{r})\right) = \vec{\nabla} \cdot \left(f(r)\,\vec{Y}^{(m)}_{JM}(\hat{r})\right) = 0$, and the third is longitudinal, $\vec{\nabla} \times \left(f(r)\,\vec{Y}^{(o)}_{JM}(\hat{r})\right) = 0$.

The Hansen set, Equation E.89, is orthnormal

$$\int d^2\Omega_{\hat{r}} \vec{Y}^{(i)^*}_{JM}(\hat{r}) \cdot \vec{Y}^{(i')}_{J'M'}(\hat{r}) = \delta_{JJ'}\delta_{MM'}\delta_{(i)(i')} \tag{E.90}$$

where i ranges over the values $((e),(m),(o))$ and complete,

$$\sum_{JM(i)} \vec{Y}^{(i)}_{JM}(\hat{r})\,\vec{Y}^{(i)^*}_{JM}(\hat{r}') = \overleftrightarrow{I}\,\delta^2\left(\Omega_{\hat{r}} - \Omega_{\hat{r}'}\right). \tag{E.91}$$

As vectors in the $((e),(m),(o))$ space, the $\vec{Y}^{(i')}_{J'M'}(\hat{r})$ are orthogonal in the sense that $\vec{Y}^{(i)^*}_{JM}(\hat{r}) \cdot \vec{Y}^{(i')}_{JM}(\hat{r}) = 0$ unless $(i') = (i)$.

These can be inverted to yield

$$\begin{aligned} \vec{Y}^{M}_{JJ-1}(\hat{r}) &= \left[\frac{J+1}{2J+1}\right]^{\frac{1}{2}} \vec{Y}^{(e)}_{JM}(\hat{r}) + \left[\frac{J}{2J+1}\right]^{\frac{1}{2}} \vec{Y}^{(o)}_{JM}(\hat{r}) \\ \vec{Y}^{M}_{JJ}(\hat{r}) &= \vec{Y}^{(m)}_{JM}(\hat{r}) \\ \vec{Y}^{M}_{JJ+1}(\hat{r}) &= -\left[\frac{J}{2J+1}\right]^{\frac{1}{2}} \vec{Y}^{(e)}_{JM}(\hat{r}) + \left[\frac{J+1}{2J+1}\right]^{\frac{1}{2}} \vec{Y}^{(o)}_{JM}(\hat{r}) \end{aligned} \tag{E.92}$$

Because of the transverse and longitudinal properties of the Hansen vector spherical harmonics, the differential operators on these are simple. Again because of Equation E.86, $\vec{\nabla}\times$ connects only terms with the same J and M.

$$\begin{aligned} \vec{\nabla} \times \left(f(r)\,\vec{Y}^{(e)}_{JM}(\hat{r})\right) &= i\frac{1}{2J+1}\left(\frac{d}{dr} + \frac{2J+1}{r}\right)\left(f(r)\,\vec{Y}^{(m)}_{JM}(\hat{r})\right) \\ \vec{\nabla} \times \left(f(r)\,\vec{Y}^{(m)}_{JM}(\hat{r})\right) &= i\frac{1}{2J+1}\left(\frac{d}{dr} + \frac{2J+1}{r}\right)\left(f(r)\,\vec{Y}^{(e)}_{JM}(\hat{r})\right) \\ \vec{\nabla} \times \left(f(r)\,\vec{Y}^{(o)}_{JM}(\hat{r})\right) &= 0 \end{aligned} \tag{E.93}$$

The divergence of the Hansen vector spherical harmonics is simply

$$\begin{aligned}
\vec{\nabla} \cdot \left(f\left(r\right) \vec{Y}_{JM}^{(e)}\left(\hat{r}\right) \right) &= 0 \\
\vec{\nabla} \cdot \left(f\left(r\right) \vec{Y}_{JM}^{(m)}\left(\hat{r}\right) \right) &= 0 \\
\vec{\nabla} \cdot \left(f\left(r\right) \vec{Y}_{JM}^{(o)}\left(\hat{r}\right) \right) &=
\end{aligned} \tag{E.94}$$

It is interesting and important to note that for $J = 0$, the only nontrivial Hansen vector spherical harmonic is the longitudinal vector spherical harmonic, $\vec{Y}_{00}^{(o)}\left(\hat{r}\right)$.