Chapter 3

String Field Theory

3.1 The String Action

The modern approach to establishing a mechanical basis for phenomena is to specify the dynamics through an action statement. Appendix A reviews the classical mechanics of the action principle. In this section, we will use this principle to find the dynamic through the Euler-Lagrange equation and to also construct some of the conserved quantities. For the stretched string, we can implement the action using the usual classical mechanics arguments of kinetic and potential energy. For the electromagnetic case though we will work the other way in identifying the action from the requirement that the Euler-Lagrange equations yield Maxwell's equations. This reversal of the application of the action principle is the current approach to theory construction.

For the massive string under tension there are two sources of energy, kinetic and potential. The kinetic is the usual kinetic energy $m\frac{mv^2}{2}$ associated with any segment of the string. Using an infinitesimal element and assuming that the mass within the element is the same regardless of the stretch of the string, the kinetic energy is

$$\begin{aligned} \mathrm{K.E.} &= \int_{-\infty}^{\infty} \left(\frac{\rho}{2}v^2\right) dx \\ &= \int_{-\infty}^{\infty} \left(\frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2\right) dx \end{aligned} \tag{3.1}$$

The potential energy is the work done stretching the string against the tension T, in order to distort it. Assuming the tension is not changed by the stretch, the work done on the string is

$$\text{P.E.} = \int_{-\infty}^{\infty} T_e \left\{ \left(\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} \right) dx - dx \right\}. \tag{3.2}$$

Using the usual definition of the Lagrangian,

$$\begin{aligned} L &= \text{K.E.} - \text{P.E.} \\ &= \int_{-\infty}^{\infty} dx \left\{ \left(\frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 \right) - T_e \left(\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} - 1 \right) \right\}. \end{aligned} \tag{3.3}$$

Appopriate to the idea of a field description, physical objects defined at every point in space, it is the practice to define a Lagrangian density,

$$\mathcal{L} \equiv \left\{ \left(\frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 \right) - T_e \left(\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} - 1 \right) \right\} \tag{3.4}$$

with correspondingly definitions for K. E. and P. E. densities. The resulting action is

$$\begin{aligned} S &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx\, \mathcal{L} \\ &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \left\{ \left(\frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 \right) \right. \\ &\qquad \left. - T_e \left(\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} - 1 \right) \right\} \end{aligned} \tag{3.5}$$

We can now adopt the usual action minimization principle to our field variables. We will do this generally so that the results will have application to any field system.

In order to visualize the process use Figure 3.1. In the particle case, the family of trajectories to be studied is between an initial event, (x_0, t_0), and a final event (x_f, t_f), see Section A.2.3. For a field, the action is computed for the family of all possible field trajectories between some initial field configuration $y_0(x, t_0)$ at time t_0 and final field configuration $y_f(x, t_f)$.

Here we will follow the Euler approach, Section A.2.8, suitably modified for field dynamics. The fundamental rule of dynamics is that the naturally occurring field trajectory is the one that is a local minimum of the action for

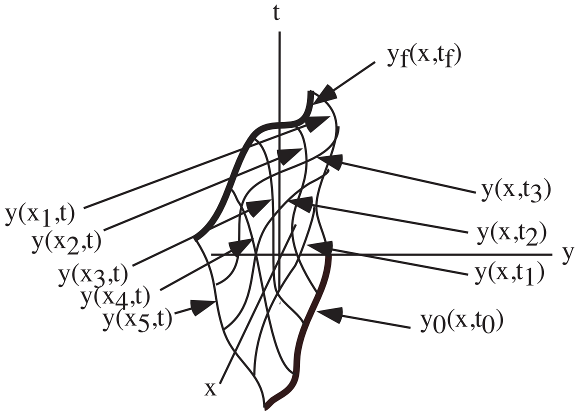

Figure 3.1: **String Fields for Action** In contrast to the case of the point particle, Figure A.4, the trajectory for the field between the initial field configuration and final field configuration is a two surface in a three space, two spatial one time. The naturally occurring field configuration is the surface configuration that minimizes the action. The dynamic for the field is found by looking at small variations about the naturally occurring configuration and requiring that the changes in the action vanish.

all field trajectories that have the same initial and final field configurations. Assuming that there exists a naturally occurring field trajectory, $\bar{y}(x,t)$, and labeling all nearby field trajectories as $y(x,t) = \bar{y}(x,t) + \delta y(x,t)$ where δy indicates the implication that the extra term is small. Since $\bar{y}(x,t_0) = y_0(x,t_0)$ and $\bar{y}(x,t_f) = y_f(x,t_f)$, we require $\delta y(x,t_0) = \delta y(x,t_f) = 0$. Some of these restrictions will be lifted later when we expand the use of the action principle.

The Lagrangian density, $\mathcal{L}$, for our string is a function of the fields in the form

$$\mathcal{L}\left(\frac{\partial y}{\partial t}(x,t), \frac{\partial y}{\partial x}(x,t)\right).$$

For the following analysis though, we will use a more general form

$$\mathcal{L}\left(\frac{\partial y}{\partial t}(x,t), \frac{\partial y}{\partial x}(x,t), y(x,t), x, t\right). \tag{3.6}$$

The Lagrangian is a function of time and a functional[1] of the field $y(x,t)$,

$$L[y(*,t)](t) = \int_{-\infty}^{\infty} dx \mathcal{L}\left(\frac{\partial y}{\partial t}(x,t), \frac{\partial y}{\partial x}(x,t), y(x,t), x, t\right). \tag{3.7}$$

The action is

$$\begin{aligned} \mathcal{S}[y(*,*)] &= \int_{-t_0}^{t_f} dt L(t) \\ &= \int_{t_0}^{t_f} dt \int_{-\infty}^{\infty} dx \mathcal{L}\left(\frac{\partial y}{\partial t}(x,t), \frac{\partial y}{\partial x}(x,t), y(x,t), x, t\right), \end{aligned} \tag{3.8}$$

which is also a functional of the field $y(x,t)$.

Rewriting the field in terms of $\bar{y}(x,t)$ and $\delta y(x,t)$, the terms entering the Lagrangian density take the form

$$\frac{\partial y}{\partial t}(x,t) = \frac{\partial \bar{y}}{\partial t}(x,t) + \frac{\partial \delta y}{\partial t}(x,t)$$

and a similar relation for the spatial derivative. The Lagrangian density is

$$\begin{aligned} \mathcal{L}\left(\frac{\partial y}{\partial t}(x,t), \frac{\partial y}{\partial x}(x,t), y(x,t), x, t\right) &= \mathcal{L}\left(\frac{\partial \bar{y}}{\partial t}(x,t) + \frac{\partial \delta y}{\partial t}(x,t),\right. \\ &\qquad \left.\frac{\partial \bar{y}}{\partial x}(x,t) + \frac{\partial \delta y}{\partial x}(x,t), \bar{y}(x,t) + \delta y(x,t), x, t\right) \\ &= \mathcal{L}\left(\frac{\partial \bar{y}}{\partial t}(x,t), \frac{\partial \bar{y}}{\partial x}(x,t), \bar{y}(x,t), x, t\right) \\ &\quad + \frac{\partial \delta y}{\partial t}\frac{\partial \mathcal{L}}{\partial \frac{\partial y}{\partial t}} + \frac{\partial \delta y}{\partial x}\frac{\partial \mathcal{L}}{\partial \frac{\partial y}{\partial x}} + \delta y \frac{\partial \mathcal{L}}{\partial y} \\ &\quad + \frac{1}{2}\left(\frac{\partial \delta y}{\partial t}\right)^2 \frac{\partial^2 \mathcal{L}}{\partial \frac{\partial y}{\partial t}^2} + \cdots, \end{aligned} \tag{3.9}$$

where the $\frac{\partial \mathcal{L}}{\partial \frac{\partial y}{\partial t}}$, $\frac{\partial \mathcal{L}}{\partial \frac{\partial y}{\partial x}}$, $\frac{\partial \mathcal{L}}{\partial y}$, and $\frac{\partial^2 \mathcal{L}}{\partial \frac{\partial y}{\partial t}^2}$ and the other higher order derivative terms are evaluated at the function $\bar{y}(x,t)$. Keeping only the first order

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[1] A functional is a quantity or function that depends on the entire course of one or more functions.The Lagrangian is called a functional of the field since it does not depend on the field in the usual function sense, a mapping of the reals into the reals. It is integrated over x and is thus a mapping of the elements of the space of functions, $y(x,t)$ for fixed t, onto the reals. This is a very rich space, bigger than $\mathbb{R}^1$, see Section A.2.2.

terms in δy and its derivatives, the change in the action for arbitrary small variations around the naturally occurring function trajectory is

$$\begin{aligned} \delta\mathcal{S} &= \int_{t_0}^{t_f} dt \int_{-\infty}^{\infty} dx \delta\mathcal{L}\left(\frac{\partial y}{\partial t}(x,t), \frac{\partial y}{\partial x}(x,t), y(x,t), x, t\right) \\ &= \int_{t_0}^{t_f} dt \int_{-\infty}^{\infty} dx \left(\frac{\partial \delta y}{\partial t}\frac{\partial\mathcal{L}}{\partial\frac{\partial y}{\partial t}} + \frac{\partial \delta y}{\partial x}\frac{\partial\mathcal{L}}{\partial\frac{\partial y}{\partial x}} + \delta y\frac{\partial\mathcal{L}}{\partial y}\right), \qquad (3.10) \end{aligned}$$

where again $\frac{\partial\mathcal{L}}{\partial\frac{\partial y}{\partial t}}$, $\frac{\partial\mathcal{L}}{\partial\frac{\partial y}{\partial x}}$, and $\frac{\partial\mathcal{L}}{\partial y}$ are evaluated at the function $\bar{y}(x,t)$. Our condition on the naturally occurring function trajectory is that, for an arbitrary small $\delta y(x,t)$, the change in the action is zero. The problem is that the variations are required to be in the small field $\delta y(x,t)$ and not the partial derivatives of the the small field. Integration by parts in t for the first term and in x for the second term,

$$\begin{aligned} \delta\mathcal{S} &= \int_{-\infty}^{\infty} dx\, \delta y(x,t)\frac{\partial\mathcal{L}}{\frac{\partial y}{\partial t}}\Big|_{t=t_0}^{t=t_f} \\ &\quad + \int_{t_0}^{t_f} dt\, \delta y(x,t)\frac{\partial\mathcal{L}}{\frac{\partial y}{\partial x}}\Big|_{x\to-\infty}^{x\to\infty} \\ &\quad - \int_{t_0}^{t_f} dt \int_{-\infty}^{\infty} dx\, \delta y(x,t)\left\{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\frac{\partial y}{\partial t}}\right) + \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\frac{\partial y}{\partial x}}\right) - \frac{\partial\mathcal{L}}{\partial y}\right\}. \\ &\qquad (3.11) \end{aligned}$$

Since $\delta y(x,t_0) = \delta y(x,t_f) = 0$ is zero for all x, the first term is zero. The second term is also zero. In this case because the family of functions being used is restricted to ones for which $y(x,t)$ vanish at $x = \pm\infty$. The condition $\delta\mathcal{S} = 0$ for an arbitrary small $\delta y(x,t)$ implies that

$$0 = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\frac{\partial y}{\partial t}}\right) + \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\frac{\partial y}{\partial x}}\right) - \frac{\partial\mathcal{L}}{\partial y} \qquad (3.12)$$

Equation 3.12 is the Euler-Lagrange equation appropriate to these fields. I can't overemphasis that since $\frac{\partial\mathcal{L}}{\partial\frac{\partial y}{\partial t}}$, $\frac{\partial\mathcal{L}}{\partial\frac{\partial y}{\partial x}}$, and $\frac{\partial\mathcal{L}}{\partial y}$ are evaluated for $y(x,t) = \bar{y}(x,t)$, this is a constraint on the function $\bar{y}(x,t)$; the naturally occurring field configuration's temporal evolution must satisfy this differential equation. There could be a concern about the neglected higher order terms. In the usual discussion, the requirement would be that with isolation of the $(\delta y)^2$ terms again using integration by parts that the coefficient be positive so that the so that this action is a minimum. Actually, the only real requirement is that the action be stationary, $\delta\mathcal{S} = 0$.

For our stretched string, the lagrangian density, Equation 3.4, the Euler-Lagrange equation is

$$\rho \frac{d\frac{\partial y}{\partial t}}{dt} = T_e \frac{d}{dx}\left\{ \frac{\frac{\partial y}{\partial x}}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}} \right\}, \tag{3.13}$$

where we have assumed ρ and T_e are constant. Equation 3.13 is not the usual wave equation for the string. In fact, it is non-linear. The usual approximation is to require small amplitude disturbances, $y(x,t) \ll 1$, and it follows that the slopes are small, $\frac{\partial y}{\partial x} \ll 1$, which allows keeping only first order terms in Equation 3.13. In addtion, all the x and t dependence comes through the fields and thus the total derivatives reduce to partial derivatives of the fields. Thus, we obtain Equation 2.3.

Of course, we could have made this approximation in the Lagrangian density and obtained Equation 2.3 from the Euler-Lagrange equation for that action directly. The more complex Lagrangian density, Equation 3.4, which leads to a non-linear Euler-Lagrange evolution equation points to the merits of Lagrange densities that are quadratic forms of the fields and their derivatives; these lead to linear differential equations. From now on we will use the simpler quadratic Lagrangian density

$$\mathcal{L} = \frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{T_e}{2}\left(\frac{\partial y}{\partial x}\right)^2. \tag{3.14}$$

Our first extension of this program is to use this action to make field systems mechanical as in a Hamiltonian theory. The procedure follows the usual development used in particle mechanics, see Section A.2.9. Identify the conjugate to the dynamical variable $y(x,t)$. The conjugate field is

$$\pi(x,t) \equiv \frac{\partial \mathcal{L}}{\partial \frac{\partial y}{\partial t}}. \tag{3.15}$$

The definition of the Hamiltonian density is

$$\mathcal{H}(x,t) \equiv \dot{y}(x,t)\frac{\partial \mathcal{L}}{\partial \frac{\partial y}{\partial t}} - \mathcal{L} \tag{3.16}$$

where $\dot{y}(x,t) = \frac{\partial y}{\partial t}$ is expressed in terms of $\pi(x,t)$ and $y(x,t)$ as found from Equation 3.15.

The Hamiltonian is

$$H(t) = \int_{-\infty}^{\infty} \mathcal{H}\, dx, \tag{3.17}$$

which is a functional of $\pi(x,t)$ and $y(x,t)$. In order to develop the evolution of the system in the Hamiltonian system of mechanics, we will have to find the suitable expression for the field Poisson bracket. This is a simple generalization of the n particle case of particle mechanics with the continuous variable x replacing the particle label n and the sums over label as an integral[2].

Specifically, the evolution of the two fields is given by the Poisson bracket equations

$$\begin{aligned} \frac{d}{dt}\left(y(x,t)\right) &= \{y(x,t), H(t)\} \\ \frac{d}{dt}\left(\pi(x,t)\right) &= \{\pi(x,t), H(t)\}. \end{aligned} \tag{3.18}$$

The Poisson bracket for the field and the conjugate is

$$\{\pi(x,t), y(x',t)\} = \delta(x-x'), \tag{3.19}$$

for all times, where $\delta(x-x')$ is the Dirac delta function, see Section B.3.

For our simple string, these are

$$\pi(x,t) = \rho \frac{\partial y}{\partial t}(x,t) \tag{3.20}$$

and

$$\mathcal{H} = \frac{1}{2\rho}\left(\pi(x,t)\right)^2 + \frac{T_e}{2}\left(\frac{\partial y}{\partial x}(x,t)\right)^2. \tag{3.21}$$

3.2 String Symmetries

Now let us look at the general category of coordinate transformations.

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[2] In order to simplify the discussion here, the relevant equations from Section A.2.9 follow. The Poisson bracket is $\{A[q_*,p_*], B[q_*,p_*]\}_{PB} = \sum_m \{\frac{\delta A}{\delta q_m}\frac{\delta B}{\delta p_m} - \frac{\delta B}{\delta q_m}\frac{\delta A}{\delta p_m}\}$ where A and B are index functionals of the q_n and p_n and $\frac{\delta A}{\delta q_m}$ is the index functional derivative. An index functional is dependent on all of the q_n and p_n in the sense $A[q_*,p_*] = \sum_n \mathcal{A}(q_n,p_n)$.