Chapter 5

The Electromagnetic Field

5.1 Introduction

Rewriting Maxwell's equations, Equations 4.14, 4.15, 4.16, and 4.17, and using the more intuitive differential forms for Div $\equiv \nabla\cdot$ and $\overrightarrow{\mathbf{Curl}} \equiv \nabla\times$ and arranged as six dynamical equations and two constraint equations, we have

$$\begin{aligned} \frac{\partial \vec{E}}{\partial t}(\vec{x},t) &= c^2\vec{\nabla}\times\vec{B}(\vec{x},t) - c^2\mu_0\vec{\mathrm{j}}(\vec{x},t) \\ \frac{\partial \vec{B}}{\partial t}(\vec{x},t) &= -\vec{\nabla}\times\vec{E}(\vec{x},t) \\ \vec{\nabla}\cdot\vec{E}(\vec{x},t) &= \frac{\rho(\vec{x},t)}{\epsilon_0} \\ \vec{\nabla}\cdot\vec{B}(\vec{x},t) &= 0. \end{aligned} \tag{5.1}$$

The first three of these equations have a name associated with their discoverer. The first is Ampère's Law with the addition of the Maxwell displacement current. The second is Faraday's Law and the third is called Coulomb's Law. The last is identified as the No Magnetic Monopoles Law as discussed below.

It is important to realize that these equations imply local conservation of the electric charge density field,

$$\frac{1}{\epsilon_0}\frac{\partial \rho}{\partial t} = \vec{\nabla}\cdot\left(\frac{\partial \vec{E}}{\partial t}\right) = \vec{\nabla}\cdot\left\{c^2\vec{\nabla}\times\vec{B} - c^2\mu_0\vec{\mathrm{j}}\right\} = -c^2\mu_0\vec{\nabla}\cdot\vec{\mathrm{j}}, \tag{5.2}$$

since $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right) = 0$. Whenever there is a scalar field and a vector field that are related as in Equation 5.2, the fields are said to carry a conserved quantity. The basis of this identification follows from the fact that, the charge, Q, within a volume V is defined as $Q \equiv \int_V d^3V \rho(\vec{x},t)$, then if the charge in the universe is conserved,

$$\begin{aligned}
\frac{d}{dt}\left(\lim_{V\to\infty}\int_V d^3V\rho(\vec{x},t)\right) &= \lim_{V\to\infty}\int_V d^3V\frac{\partial}{\partial t}\rho(\vec{x},t) \\
&= -\lim_{V\to\infty}\int_V d^3V\vec{\nabla}\cdot\vec{\mathrm{j}}(\vec{x},t) \\
&= -\lim_{V\to\infty}\oint_{\partial V} d^2s\,\vec{j}(\vec{x},t)\cdot\hat{s} \\
&= 0. \qquad (5.3)
\end{aligned}$$

The first line is just the identification that all the time dependence of the charge is contained in the density's time dependence. The second step is the substitution of the divergence of the associated charged for the time rate of change of the density. Finally, Gauss' s theorem[1] leads to the integrals over the surrounding surfaces which are now at infinity. If the currents all vanish at spatial infinity, this term is zero. It follows that, even if the volume is finite, the third line would read that the time rate of change in the volume manifests as the flow through the surface surrounding that volume. Also note that we have now generalized our notation as regards geometric structures. The designation ∂V is the surface bounding V. This same notation is applied to any geometric structure in any number of dimensions. Another case that will be important for us is the ∂s is the curve bounding a surface area $\vec{s}$. An important observation is that the bounding sets have no boundary, $\partial\partial V = 0$. This will be important when we try to develop a more geometric interpretation of Maxwell's equations, see Section 5.3, and Appendix E

There is one further issue that is essential to a complete dynamic description and that is the reaction of the charges to the presence of the fields. There is a local force density, $\vec{f}(\vec{x},t)$, defined as

$$\vec{f}(\vec{x},t) \equiv \rho(\vec{x},t)\vec{E}(\vec{x},t) + \vec{j}(\vec{x},t)\times\vec{B}(\vec{x},t). \qquad (5.5)$$

[1] Gauss's theorem is equivalence between the volume integral of a divergence of a vector field and the integral over the bounding surface of the vector field along the direction of the surface elements,

$$\int_V d^3V\vec{\nabla}\cdot\vec{\mathrm{V}}(\vec{x},t) = \oint_{\partial V} d^2s\,\vec{V}(\vec{x},t)\cdot\hat{s}. \qquad (5.4)$$

This reduces to the usual Lorentz force on a particle with charge q,

$$\vec{F}_q = q\left\{\vec{E}(\vec{x}_0(t),t) + \vec{v}\times\vec{B}(\vec{x}_0(t),t)\right\}. \tag{5.6}$$

The point charge can be described as a charge density field as $\rho(\vec{x},t) = q\delta^3(\vec{x}-\vec{x}_0(t))$ where $\vec{x}_0(t)$ is the position of the charge at time t. It is interesting to note that the current field is a convection field of the charge density, i.e. the fundamental entity is the charge and the current is seen as charge in motion; $\vec{j}\,(\vec{x},t) = q\frac{d\vec{x}_0(t)}{dt}\delta^3(\vec{x}-\vec{x}_0(t))$. This is contrary to the usual definition of charge. Charge is defined as the time integral of a current. Some implications of this are discussed in Section 8.2 and 6.3. The conservation law, Equation 5.3, interpreted in terms of the point charge description above implies that the charge elements, the electron, are indestructible and cannot be produced by any process involving this dynamic.

With the force equation in the form of Equation 5.6, we realize that the usual language to describe fields $\vec{\mathbf{E}}(\vec{x}_0(t),t)$ and $\vec{\mathbf{B}}(\vec{x}_0(t),t)$ are not simply described as vector fields. Force is a vector and thus $\vec{\mathbf{E}}(\vec{x}_0(t),t)$ is a simple vector field. On the other hand, $\vec{\mathbf{B}}(\vec{x}_0(t),t)$ must be a pseudo-vector[2]; the vector product of two vectors produces a pseudo-vector. This differentiation is due to the fact that $\vec{\mathbf{B}}$ actually transforms like an area and in three space an area transforms under rotations like a vector[3] but oppositely from a vector under inversions. This structure of the electric and magnetic fields is clarified when geometric concepts are used to describe electromagnetic phenomena. This idea will be developed more fully in Section 5.3, and in Appendix E.

Another important aspect of these equations is the absence of a magnetic charge. There is no reason why a fundamental magnetic charge, m, its density, $\rho_m(\vec{x},t) = m\delta^3(\vec{x}-\vec{x}_0(t))$, and its current, $\vec{j}_m(\vec{x},t) = m\frac{d\vec{x}_0(t)}{dt}\delta^3(\vec{x}-\vec{x}_0(t))$ do not exist. Their existence would lead to a more symmetric form of the Maxwell field theory. It will also require modifications of the Lorentz force. The realization of this possibility will color our discussion of the source free fields and the non-relativistic limit of the Maxwell field system, see Section 8.2 and our understanding of the geometric nature of the electromagnetic fields, Section 5.3.

It is important to realize also the importance of the linearity of this system. Maxwell's equations are linear in all the fields including the source

[2]A pseudo-vector is a three component element that transforms under rotations as a vector but does not change sign under inversions.

[3]This is based on the fact that three space has an invariant three tensor, $\epsilon^{\alpha\beta\gamma}$, the alternating or Levi-Civita symbol, see Section D

fields. This implies the superposition of all partial solutions. For example, source free solutions can be added to solutions with sources. This has important implications on how light interacts with matter. It also leads to the linearity of sources in the description of constructs that are also linear in the field strengths such as magnetic flux linkages that are the basis of inductance between circuit elements.

It is also worthwhile displaying the Maxwell equations in integral form. These are found by inverting the definitions of Div or $\vec{\nabla}\cdot$ and $\overrightarrow{\mathbf{Curl}}$ or $\vec{\nabla}\times$. These are

$$\begin{aligned}
\frac{\partial}{\partial t}\left\{\int_s d^2s\,\hat{s}\cdot\vec{\mathbf{E}}\right\} &= c^2\oint_{\partial s} dz\frac{d\vec{r}}{dz}\cdot\vec{\mathbf{B}} - c^2\mu_0\left\{\int_s d^2s\,\hat{s}\cdot\vec{\mathbf{j}}\right\} && (5.7)\\
\frac{\partial}{\partial t}\left\{\int_s d^2s\,\hat{s}\cdot\vec{\mathbf{B}}\right\} &= -\oint_{\partial s} dz\frac{d\vec{r}}{dz}\cdot\vec{\mathbf{E}}\hat{s} && (5.8)\\
\oint_{\partial V} d^2\,s\,\hat{s}\cdot\vec{\mathbf{E}} &= \frac{1}{\epsilon_0}\int_V d^3V\rho = \frac{Q_{inside\ V}}{\epsilon_0} && (5.9)\\
\oint_s d^2\,s\,\hat{s}\cdot\vec{\mathbf{B}} &= 0. && (5.10)
\end{aligned}$$

These will take a more recognizable form when we define some terms. The flux of a vector field through a surface is $\int_s d^2s\,\hat{s}\cdot\vec{\mathbf{V}} \equiv \Phi_{\vec{V}}(s)$. The integral around a closed loop of the electric field is called the electromotive force and designated $\mathcal{E}mf(s) \equiv \oint_{\partial s} dz\frac{d\vec{r}}{dz}\cdot\vec{\mathbf{E}}$ where ∂s is the loop surrounding the surface s, with a corresponding designation for the magnetic term or magnetomotive force, $\mathcal{B}mf(s) \equiv \oint_{\partial s} dz\frac{d\vec{r}}{dz}\cdot\vec{\mathbf{B}}$. This later term does not have the same relevance as the $\mathcal{E}mf$ because work is done on a charged particle by the electric force whereas it is not by the magnetic force. Equations 5.7 and 5.8 become

$$\begin{aligned}
\frac{d}{dt}\Phi_{\vec{E}}(s) &= c^2\mathcal{B}mf(s) - c^2\mu_0 i(s) && (5.11)\\
\frac{d}{dt}\Phi_{\vec{B}}(s) &= -\mathcal{E}mf(s), && (5.12)
\end{aligned}$$

where $i(s)$ is the current or flux of charge through the area s.

It is very useful to classify any vector field into transverse and longitudinal components. A vector field, $\vec{V}(\vec{r},t)$, that has zero divergence everywhere, $\vec{\nabla}\cdot\vec{V}(\vec{r},t) = 0$, is defined as transverse or solenoidal. A vector field that has zero curl everywhere, $\vec{\nabla}\times\vec{V}(\vec{r},t) = 0$, is called longitudinal or irrotational. The separation of the field into these two components is best understood in wave-number or Fourier space, see Appendix B.

The fields in wave-number or $\vec{k}$ space are defined as

$$\vec{\mathcal{E}}(\vec{k},t)=\frac{1}{\left(\sqrt{2\pi}\,\right)^3}\int_{-\infty}^{\infty}d^3\vec{r}\,\vec{E}(\vec{r},t)e^{-i\vec{k}\cdot\vec{r}} \tag{5.13}$$

and the field is

$$\vec{E}\,(\vec{r},t)=\frac{1}{\left(\sqrt{2\pi}\,\right)^3}\int_{-\infty}^{\infty}d^3\vec{k}\,\vec{\mathcal{E}}(\vec{k},t)e^{i\vec{k}\cdot\vec{r}} \tag{5.14}$$

with a similar set of definitions for the magnetic field and any other field variable. It is worth noting that Equation 5.14 goes beyond the usual three dimensional Fourier transform of a scalar field, see Section B.4. The coordinate basis for the vector fields are chosen to be the same in configuration and k-space. For example, if the basis for the expansion of the configuration space fields is in cylindrical coordinates so is the k-space basis. Also the directions of the basis vectors are the same. For the example of the cylindrical coordinate configuration space, (ρ,θ,z), the k space basis vectors are

In $\vec{k}$ space, the Maxwell's dynamical equations become

$$\begin{aligned}\frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathcal{E}}(\vec{k},t) &= -i\vec{k}\times\vec{\mathcal{B}}(\vec{k},t)-\mu_0\vec{\mathcal{J}}(\vec{k},t)\\ \frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t) &= i\vec{k}\times\vec{\mathcal{E}}(\vec{k},t),\end{aligned} \tag{5.15}$$

and the constraint equations become

$$\begin{aligned}\vec{k}\cdot\vec{\mathcal{E}}(\vec{k},t) &= \frac{\rho(\vec{k},t)}{\epsilon_0}\\ \vec{k}\cdot\vec{\mathcal{B}}(\vec{r},t) &= 0.\end{aligned} \tag{5.16}$$

In $\vec{k}$ space the condition of transversality is simply $\vec{k}\cdot\vec{\mathcal{V}}(\vec{k},t)$=0. Longitudinal vector fields satisfy $\vec{k}\times\vec{\mathcal{V}}(\vec{k},t)=0$. For any vector $\vec{k}$ in wave-number space, we can construct a complete right handed triplet of unit vectors $\hat{k}\equiv\frac{\vec{k}}{k}$, and $\hat{\epsilon}^{(1)}(\hat{k})$, and $\hat{\epsilon}^{(2)}(\hat{k})\equiv\hat{k}\times\epsilon^{(1)}(\hat{k})$ where $k\equiv\|\vec{k}\|$ and $\hat{\epsilon}^{(1)}(\hat{k})$ is a unit vector in the plane orthogonal to $\vec{k}$. It is interesting to point out that the ambiguity in the choice of the direction of $\hat{\epsilon}^{(1)}(\hat{k})$ is the reason for the spin one nature of the photon, see Section **??**.

Using this basis, any vector field expressed in $\vec{k}$ space can be expanded into its longitudinal and transverse parts,

$$\vec{\mathcal{V}}=\mathcal{V}^L\hat{k}+\mathcal{V}^T_{(1)}\hat{\epsilon}^{(1)}(\hat{k})+\mathcal{V}^T_{(2)}\hat{\epsilon}^{(2)}(\hat{k}), \tag{5.17}$$

the component along $\hat{k}$ being longitudinal, $\vec{k}\times\mathcal{V}^L\hat{k}=0$, and the component in the plane of $\hat{\epsilon}^{(1)}(\hat{k})$ and $\hat{\epsilon}^{(2)}(\hat{k})$ being transverse, $\vec{k}\cdot\left(\mathcal{V}^T_{(1)}\hat{\epsilon}^{(1)}(\hat{k})+\mathcal{V}^T_{(2)}\hat{\epsilon}^{(2)}(\hat{k})\right)=0$. In fact in $\vec{k}$ space, the projector for the longitudinal part of any vector field is simply

$$\vec{\mathcal{V}}_i^L(\vec{k},t)\equiv\sum_{j=1,2,3}\frac{k_ik_j}{k^2}\vec{\mathcal{V}}_j(\vec{k},t)=(\hat{k}\cdot\vec{\mathcal{V}})\hat{k}_i \tag{5.18}$$

and, since the basis for longitudinal and transverse separations is complete, $\vec{\mathcal{V}}^T(\vec{k},t)=\vec{\mathcal{V}}(\vec{k},t)-\vec{\mathcal{V}}^L(\vec{k},t)$ or

$$\vec{\mathcal{V}}_i^T(\vec{k},t)=\sum_{j=1,2,3}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\vec{\mathcal{V}}_j(\vec{k},t) \tag{5.19}$$

is the transverse part. Obviously, $\vec{k}\cdot\vec{\mathcal{V}}^T(\vec{k},t)=0$ for all $\vec{\mathcal{V}}^T(\vec{k},t)$ and $\vec{k}\times\vec{\mathcal{V}}^L(\vec{k},t)=0$ for all $\vec{\mathcal{V}}^L(\vec{k},t)$.

Also given any longitudinal field $\vec{V}^L(\vec{x}.t)$ it can be associated with a scalar field, $\phi(\vec{x},t)$, in the sense that, since $\vec{\nabla}\times\vec{\nabla}\phi(\vec{x},t)=0$ everywhere, the quantity $\vec{\nabla}\phi(\vec{x},t)$ is a longitudinal vector field. These two longitudinal vector fields can be equated if Since the vector field is longitudinal, $\vec{k}\phi(\vec{k},t)$

The constraint equation for the magnetic field, Equation 5.16 requires that the magnetic field be transverse. Elements such as $\vec{k}\times\vec{\mathcal{E}}$ or $\vec{k}\times\vec{\mathcal{B}}$ are also transverse. Projecting the dynamical equations of the $\vec{k}$ fields in Equations 5.15 ,

$$\vec{\mathcal{E}}(\vec{k},t)=\vec{\mathcal{E}}^T(\vec{k},t)+\vec{\mathcal{E}}^L(\vec{k},t) \tag{5.20}$$

with similar definitions for all the other vector fields, Maxwell's equations in $\vec{k}$ space become

$$\begin{aligned}\frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathcal{E}^L}(\vec{k},t) &= -\mu_0\mathcal{J}^L(\vec{k},t)\\ \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathcal{E}^T}(\vec{k},t) &= -i\vec{k}\times\vec{\mathcal{B}^T}(\vec{k},t)-\mu_0\mathcal{J}^T(\vec{k},t)\\ \frac{\partial}{\partial t}\vec{\mathcal{B}^L}(\vec{k},t) &= 0\\ \frac{\partial}{\partial t}\vec{\mathcal{B}^T}(\vec{k},t) &= i\vec{k}\times\vec{\mathcal{E}^T}(\vec{k},t),\end{aligned} \tag{5.21}$$

and the constraint equations become

$$\begin{aligned}\vec{k}\cdot\vec{\mathcal{E}^L}(\vec{k},t) &= \frac{\rho(\vec{k},t)}{\epsilon_0}\\ \vec{k}\cdot\vec{\mathcal{B}^L}(\vec{r},t) &= 0.\end{aligned} \tag{5.22}$$

along with the defining equations

$$\begin{aligned}
\vec{k}\cdot\vec{\mathcal{E}}^{T}(\vec{k},t) &= 0\\
\vec{k}\times\vec{\mathcal{E}}^{L}(\vec{k},t) &= 0\\
\vec{k}\cdot\vec{\mathcal{B}}^{T}(\vec{k},t) &= 0\\
\vec{k}\times\vec{\mathcal{B}}^{L}(\vec{k},t) &= 0\\
\vec{k}\cdot\vec{\mathcal{J}}^{T}(\vec{k},t) &= 0\\
\vec{k}\times\vec{\mathcal{J}}^{L}(\vec{k},t) &= 0.
\end{aligned} \tag{5.23}$$

In addition, the charge conservation condition, Equation 5.2, becomes

$$\frac{\partial\rho}{\partial t}(\vec{k},t) + i\vec{k}\cdot\vec{\mathcal{J}}^{L}(\vec{k},t) = 0. \tag{5.24}$$

Although this is a formidable dynamical system to analyze, it is clear that there is a decoupling of the transverse and the longitudinal fields. This insight will help in our interpretation electromagnetic phenomena.

Also the form of Equation 5.17 indicates that given any scalar field, $\phi(\vec{x},t)$, the quantity $\vec{\nabla}\phi(\vec{x},t)$ is a longitudinal vector field since $\vec{k}\phi(\vec{k},t)$

In keeping with the spirit of a field theory description of electromagnetic phenomena, it is appropriate to first investigate the details of the source free fields. This is the usual approach when discussing fields and the addition of sources is treated only after the free fields are fully developed. For E & M, this is especially important because of the generally inadequate appreciation of the reality of the field degrees of freedom. Many students do not seem to think that there are phenomena in which the free electromagnetic fields, situations with no currents or charges, play any role in nature. There is a habit to attach the fields to the charges or currents. This, of course, violates the fundamental precepts of local field theory as discussed in Section 2.1. As we will see the free fields are the basis for our understanding of the properties of radiation. The phenomena associated with the thermodynamics of black bodies which, of course, is the basis of quantum mechanics and ultimately the existence of the photon is a case in which the field degrees of freedom operate without any role to be played by charges and currents. Remember the heat capacity of the black body cavity scales as the volume of the cavity and there are no charges in the cavity. Of course, the concept of thermal equilibrium and the coupling between thermal bath of matter and and a thermal bath of radiation ultimately requires that the charges and currents play a role connecting the different degrees of freedom but that comes later.

Therefore, this approach will introduce the fields as mechanical systems with all their properties following from standard classical mechanics. We will develop the kinematics of the field system and derive an action the yields the correct Maxwell equations. From this action all the important properties of the fields will be developed. Only after there is a complete understanding of the electromagnetic fields will the sources be introduced as separate physical entities.

In addition, in the first pass at understanding the electromagnetic field, we will not be concerned with the electromagnetic field in bounded spaces. As with the string, we treat the infinite space system before adding boundaries. We will add the effects of the boundaries after the free space fields are fully understood.

5.2 Two Time Scales

5.2.1 Static

There are three time scales that play an important role in any treatment of electromagnetic phenomena. The simplest is the static limit; there is no time dependence. Of course, this is an approximation for any dynamic system, particularly one as complex as the electromagnetic field with sources, since the charges, currents, and field strengths have to be configured into their stationary arrangement. There will generally be non-trivial situations that somehow have to settle out and allow the system to settle down to the described configuration.

The Maxwell system reduced to the static case is

$$\vec{\nabla} \times \vec{B}(\vec{x}) = \mu_0 \vec{\mathrm{j}}(\vec{x}) \tag{5.25}$$

$$\vec{\nabla} \times \vec{E}(\vec{x}) = 0 \tag{5.26}$$

$$\vec{\nabla} \cdot \vec{E}(\vec{x}) = \frac{\rho(\vec{x})}{\epsilon_0} \tag{5.27}$$

$$\vec{\nabla} \cdot \vec{B}(\vec{x}) = 0. \tag{5.28}$$

Also, the first equation requires

$$\vec{\nabla} \cdot \vec{j}(\vec{x}) = 0. \tag{5.29}$$

These equations yield that, as usual, $\vec{B}(\vec{x})$ is transverse, but also $\vec{E}(\vec{x})$ is longitudinal, and $\vec{j}(\vec{x})$ is transverse.

In this limit, the electric and magnetic fields decouple and the sources are no longer coupled so that it is the usual approach is examine magnetostatics and electrostatics as separate subjects.

Electrostatics

In this case, if the system possesses sufficient symmetry, the solution for the electric field can be found by direct application of Gauss's law in integral form, see Equation 5.9, by picking a volume with a bounding surface such that the integral of the electric field averaged over the surface produces usable information about the value of the electric field everyshere. As an example consider the case of a system possessing translation symmetry in one direction, called $\hat{z}$, and rotational symmetry about the z axis. Using a cylindrical coordinate system, (ρ, φ, z), because of these symmetry requirements, the electric field cannot depend on φ or z and the longitudinal nature of the electric field, Equation 5.26, requires that $\frac{\partial E_z}{\partial \rho} = 0$ and $\frac{\partial(\rho E_\varphi)}{\partial \rho} = 0$[4], which requires that the only none vanishing component of the electric field is $\vec{E}(\vec{x}) = E_\rho(\rho)\hat{\rho}$.

For the example of the electric field outside a cylindrical charge distribution, a surface in the shape of a can in the form of a cylinder with its axis of symmetry on the z axis and planar top and bottom yields the solution process:

$$\begin{aligned} \oint_{\partial V} \vec{E} \cdot \hat{s} d^2 s &= \int_V \frac{\rho(\rho)}{\epsilon_0} \rho d\rho d\varphi dz \\ \int_{side} \vec{E} \cdot \hat{\rho} \rho d\varphi dz &= \frac{Q_{\text{enclosed}}}{\epsilon_0} \\ 2\pi\rho h E_\rho(\rho) &= \frac{Q_{\text{enclosed}}}{\epsilon_0} \end{aligned} \tag{5.31}$$

where h is the height of the can. Thus for this case,

$$\vec{E}\left(\vec{x}\right) = \frac{\lambda_{\text{enclosed}}}{2\pi\epsilon_0\rho}\hat{\rho} \tag{5.32}$$

where $\lambda_{\text{enclosed}}$ is the charge per unit length along the $z$axis inside a cylinder of radius ρ.

The more general approach to the solution of the electrostatic problems is to take advantage of the longitudinal nature of the electric field. The

[4]This condition only requires that $\rho E_\varphi =$ constant. The integral form of the curl for a closed circular path at some ρ_0 then requires

$$\begin{aligned} \oint_{\rho_0} \vec{E} \cdot \hat{\varphi}\, d\varphi &= 0 \\ \Rightarrow 2\pi E_\varphi\left(\rho_0\right) &= 0 \end{aligned} \tag{5.30}$$

which implies that the constant is zero.

field $\vec{\nabla}\phi(\vec{x})$ for any scalar field $\phi(\vec{x})$ is longitudinal. The electric field can be constructed from the identification

$$\vec{E}(\vec{x}) = -\vec{\nabla}\phi(\vec{x}) \tag{5.33}$$

with the condition that

$$\vec{\nabla}\cdot\vec{\nabla}\phi(\vec{x}) = \vec{\nabla}^2\phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0} \tag{5.34}$$

For the case of the infinite domain, the simplest approach is to use the Fourier transform to solve Equation 5.34

$$\vec{k}^2\mathcal{P}(\vec{k}) = \frac{\mathcal{R}(\vec{k})}{\epsilon_0} \tag{5.35}$$

where $\mathcal{P}(\vec{k})$ is the Fourier transform of $\phi(\vec{x})$ and $\mathcal{R}(\vec{k})$ is the Fourier transform of $\rho(\vec{x})$. The solution is simply that

$$\mathcal{P}(\vec{k}) = \frac{1}{k^2}\frac{\mathcal{R}(\vec{k})}{\epsilon_0}. \tag{5.36}$$

Using the convolution theorem, Equation B.20, the potential is the convolution of the Fourier transforms of $\frac{1}{k^2}$ and $\frac{\mathcal{R}(\vec{k})}{\epsilon_0}$ or

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}}\frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}d^3\vec{x}\,' \tag{5.37}$$

Thus given a localized charge distribution and the condition that the scalar field be Fourier transformable, $\lim_{\vec{x}\to\infty}\phi(\vec{x})\to 0$ sufficiently rapidly, yields a solution everywhere.

For localized charge distributions and the field point outside the charged space in an unbounded space, this analyze usually goes one step further. Using Equation F.56, Equation 5.37 can be written

$$\phi(\vec{x}) = \frac{1}{\epsilon_0}\int\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r'^l}{r^{l+1}}Y_l^m(\hat{r})\,Y_l^{m^*}(\hat{r}')\,\rho(\vec{r}\,')d^3\vec{r}\,' \tag{5.38}$$

where we have used the fact that in this case $r_<(r,r') = r'$ and $r_>(r,r') = r$. We can reorganize this result into

$$\phi(\vec{x}) = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{q_{lm}}{r^{l+1}}Y_l^m(\hat{r})\,. \tag{5.39}$$

where

$$q_{lm} \equiv \int r\,'^l Y_l^{m^*}(\hat{r}\,')\,\rho(\vec{r}\,')d^3\vec{r}\,'. \tag{5.40}$$

The q_{lm} are called the multipole moments of the charge distribution.

Magnetostatics

The case of magnetostatics proceeds similarly to the electrostatics case. The most straight forward way find a solution to parts of Equations 5.28 is to use Stokes theorem in integral form. Again this is possible only in a situation with enough symmetry to allow the path averaged field to yield information about the field everywhere.

5.2.2 Quasistatic

5.3 Geometric Description of the Maxwell System

The system of equations that articulate the behavior of charged particles and the electromagnetic fields as given in Equations 5.1 through 5.5 are written in vector notation. In a Euclidean three space, this is a compact and interpretable scheme although even in its simplest applications there are indications that a richer form serves better, see Footnote 3 in Section 5.1.

In Appendix E, we describe the development of a mathematical description of a calculus on manifolds that recognizes larger categories of important geometric constructs.