Chapter 6

Source Free Electromagnetic Fields

Maxwell's equations, Equations 5.1, with the source terms removed are relatively simple. There are six temporal evolution equations

$$\begin{aligned} \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) &= \vec{\nabla}\times\vec{B}(\vec{r},t) \\ \frac{\partial}{\partial t}\vec{B}(\vec{r},t) &= -\vec{\nabla}\times\vec{E}(\vec{r},t), \end{aligned} \tag{6.1}$$

and two constraint equations

$$\begin{aligned} \vec{\nabla}\cdot\vec{E}(\vec{r},t) &= 0 \\ \vec{\nabla}\cdot\vec{B}(\vec{r},t) &= 0. \end{aligned} \tag{6.2}$$

These constraints require that the source free case has only transverse solutions, $\vec{E}^T$ and $\vec{B}^T$. We will not carry the designation of the transverse nature of the field in the following but the reader should keep this condition in mind.

This is a set of first order in time coupled field equations much like the string, Equation 2.59, with the addition of constraints. Similar to that case, a pair of second order decoupled equations can be found and these are the usual wave equation for both the fields,

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t) - \vec{\nabla}^2\vec{E}(\vec{r},t) = 0 \tag{6.3}$$

and

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B}(\vec{r},t) - \vec{\nabla}^2\vec{B}(\vec{r},t) = 0. \tag{6.4}$$

Thus, all the solutions of this field system are travelers with velocity $\pm c$. This, of course, is the usual definition of radiation. There are complications well beyond those of a string because of the three spatial dimensions and the vector nature of the fields but otherwise all the intuition developed there carries over to this system, see Section 2.2. As with the string, the system is hyperbolic and, as we will see shortly, other than the phase relations between $\vec{E}$ and $\vec{B}$, it behaves much like a string when described using the displacement/velocity or Hamiltonian description, Equation 2.59. It is also worth pointing out that the source free electromagnetic field system has only one dimensional parameter, the speed of light c. This means that this system has no intrinsic length or time scales but does have an intrinsic velocity. Thus the length and time scales for any field configuration are connected; a disturbance of length scale L has a time scale $\frac{L}{c}$.

There is a special quadratic form for the source free fields that provides special constraints and interpretation. The quantity

$$\rho_e(\vec{x},t) \equiv \frac{1}{2}\left(\vec{E}^2(\vec{x},t) + c^2\vec{B}^2(\vec{x},t)\right) \tag{6.5}$$

is locally conserved in the same sense as the charge, Equation 5.3. When the time rate of change of $\rho_e(\vec{x},t)$ is examined

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\left(\vec{E}^2(\vec{x},t) + c^2\vec{B}^2(\vec{x},t)\right) &= \left(\left(\frac{\partial}{\partial t}\vec{E}\right)\cdot\vec{E} - c^2\vec{B}\cdot\left(\frac{\partial}{\partial t}\vec{B}\right)\right) \\
&= \left(c^2\left(\vec{\nabla}\times\vec{B}\right)\cdot\vec{E} - c^2\vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right)\right) \\
&= c^2\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right). \qquad (6.6)
\end{aligned}$$

Thus $\vec{j}_e(\vec{x},t) \equiv -c^2\left(\vec{E}(\vec{x},t)\times\vec{B}(\vec{x},t)\right)$ is a local flow current that balances the change in the density[1], $\frac{\partial\rho_e}{\partial t} + \vec{\nabla}\cdot\vec{j}_e = 0$. Once we develop the action for these fields, we will derive this local density from the Noether's construction that develops from time translation symmetry, see Sections A.5.4 and 6.3 . Obviously, other continuous symmetries will lead to other local conservation densities. Equations 6.1 and 6.2 also enjoy rotational and time and space translational symmetries.

[1]In the presence of sources this relation becomes

$$\frac{\partial\rho_e}{\partial t} + \vec{\nabla}\cdot\vec{j}_e = -c^2\mu_0\vec{j}(\vec{x},t)\cdot\vec{E}(\vec{x},t). \tag{6.7}$$

There is a source for this otherwise conserved density. In the usual interpretation, power is transferred to the charges by the electric field, we can identify this density as the energy density.

There are other symmetries of this dynamic. The substitution

$$\begin{aligned}\vec{E}'(\vec{x},t) &= c\vec{B}(\vec{x},t)\\ \vec{B}'(\vec{x},t) &= -\frac{\vec{E}(\vec{x},t)}{c}\end{aligned} \tag{6.8}$$

leaves Equations 6.1 through 6.6 unchanged.

In addition, Equations 6.3 and 6.4 are wave equations and, as in the string example, Section 2.3.5, are unchanged by the linear coordinate transformation, the Lorentz transformations, that preserves the form $c^2t'^2 - x'^2 = c^2t^2 - x^2$, see Section 2.3.5 and G.3. This symmetry is true for Equations 6.3 and 6.4 also for the electric and magnetic fields in each component and, since they linear, they remain true for any linear transformation of the fields. The Maxwell dynamic in the form of Equations 6.1 and 6.2 is not symmetric without also making a linear transformation in the fields. This problem is reviewed in Section 8.2.

Similarly, Equations 6.1 and 6.2 also have other discrete symmetries. Spatial reflection, $\vec{x} \to -\vec{x}$, also requires that $\vec{E}'(\vec{x}',t) = -\vec{E}(-\vec{x},t)$ and $\vec{B}'(\vec{x}',t) = \vec{B}(-\vec{x},t)$; that $\vec{E}(\vec{x},t)$ be a vector field and $\vec{B}(\vec{x},t)$ be a pseudovector field or better said a two form, an area, see Section 5.3 . Time reversal invariance

6.1 Source Free Fields in $\vec{k}$ Space

The kinematics and dynamics of the source free Maxwell fields is best understood when the spatial labels are transformed into Fourier or wave-number space, $\vec{k}$ space. The properties and the nature of these transfoms and the closely related generalized functions are treated in some detail in Appendix B. It must be emphasized that the examination of the Maxwell field in $\vec{k}$ space is appropriate only to the free space field properties. The bounded system will require special considerations and be treated later, see Chapter 9.

For the case of the field in unbounded space, these transforms are defined as

$$\begin{aligned}\vec{\mathcal{E}}(\vec{k},t) &= \frac{1}{\left(\sqrt{2\pi}\right)^3}\int_{-\infty}^{\infty} d^3\vec{r}\,\vec{E}(\vec{r},t)e^{-i\vec{k}\cdot\vec{r}}\\ \vec{E}\,(\vec{r},t) &= \frac{1}{\left(\sqrt{2\pi}\right)^3}\int_{-\infty}^{\infty} d^3\vec{k}\,\vec{\mathcal{E}}(\vec{k},t)e^{i\vec{k}\cdot\vec{r}}\end{aligned} \tag{6.9}$$

with a similar set of definitions for the magnetic field and any other field variable.

In $\vec{k}$ space, the Maxwell's dynamical equations become

$$\begin{aligned} \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathcal{E}}(\vec{k},t) &= -i\vec{k}\times\vec{\mathcal{B}}(\vec{k},t) \\ \frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t) &= i\vec{k}\times\vec{\mathcal{E}}(\vec{k},t), \end{aligned} \tag{6.10}$$

and the constraint equations become

$$\begin{aligned} \vec{k}\cdot\vec{\mathcal{E}}(\vec{k},t) &= 0 \\ \vec{k}\cdot\vec{\mathcal{B}}(\vec{r},t) &= 0. \end{aligned} \tag{6.11}$$

Equations 6.11 require that the source free electromagnetic fields be transverse. This is consistent with the dynamical equations, Equations 6.10, since the longitudinal projector can be used on Equation 6.10 to maintain the constraint for all times,

$$\begin{aligned} \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathcal{E}}^L(\vec{k},t) &= 0 \\ \frac{\partial}{\partial t}\vec{\mathcal{B}}^L(\vec{k},t) &= 0. \end{aligned} \tag{6.12}$$

It is important to realize that since the dynamical equations, Equations 6.10, involve only the transverse fields which are only two dimensional, see Equation 5.17, there are only four equations instead of six as the notation may indicate. Also these fields in $\vec{k}$ space couple only fields of the same $\vec{k}$ and, thus, for each $\vec{k}$ this is a particularly simple mechanical system; the evolution of the field for each $\vec{k}$ develops independently. Expanding the transverse fields in terms of the $\hat{\epsilon}^{(1)}(\hat{k})$ and $\hat{\epsilon}^{(2)}(\hat{k})$ basis,

$$\vec{\mathcal{E}}^T(\vec{k},t) = \mathcal{E}^T_{(1)}(\vec{k},t)\hat{\epsilon}^{(1)}(\hat{k}) + \mathcal{E}^T_{(2)}(\vec{k},t)\hat{\epsilon}^{(2)}(\hat{k}), \tag{6.13}$$

with a similar expansion for $\vec{\mathcal{B}}^T(\vec{k},t)$.

The dynamic for this system is

$$\begin{aligned} \frac{1}{c^2}\frac{\partial\mathcal{E}^T_{(1)}}{\partial t}(\vec{k},t) &= ik\mathcal{B}^T_{(2)}(\vec{k},t) \\ \frac{\partial\mathcal{B}^T_{(2)}}{\partial t}(\vec{k},t) &= ik\mathcal{E}^T_{(1)}(\vec{k},t) \\ \frac{1}{c^2}\frac{\partial\mathcal{E}^T_{(2)}}{\partial t}(\vec{k},t) &= -ik\mathcal{B}^T_{(1)}(\vec{k},t) \\ \frac{\partial\mathcal{B}^T_{(1)}}{\partial t}(\vec{k},t) &= -ik\mathcal{E}^T_{(2)}(\vec{k},t). \end{aligned} \tag{6.14}$$

Thus the four dynamical equations for the fields decouple even further into a pair of coupled first order differential equations with the $\mathcal{E}^T_{(1)}(\vec{k},t)$ and $\mathcal{B}^T_{(2)}(\vec{k},t)$ coupled and $\mathcal{E}^T_{(2)}(\vec{k},t)$ and $\mathcal{B}^T_{(1)}(\vec{k},t)$ coupled and each pair independent of the other. Calling these the $\hat{\epsilon}^{(1)}$ and $\hat{\epsilon}^{(2)}$ field systems respectively. These two coupled one dimensional fields are the only non-trivial fields that are possible for the source free case of the electromagnetic fields.

The general solutions of these equations are those of two independent simple harmonic oscillators both with radian frequency $\omega = \pm kc$ or

$$\begin{aligned} \mathcal{E}^T_{(1)}(\vec{k},t) &= \mathcal{E}^T_{(1)}(\vec{k},0)\cos(kct) + ic\mathcal{B}^T_{(2)}(\vec{k},0)\sin(kct) && (6.15)\\ c\mathcal{B}^T_{(2)}(\vec{k},t) &= \frac{-i}{kc}\frac{\partial}{\partial t}\mathcal{E}^T_{(1)}(\vec{k},t) && (6.16)\\ &= c\mathcal{B}^T_{(2)}(\vec{k},0)\cos(kct) + i\mathcal{E}^T_{(1)}(\vec{k},0)\sin(kct), && (6.17) \end{aligned}$$

with a similar pair of solutions for the $\mathcal{E}^T_{(2)}(\vec{k},t)$ and $\mathcal{B}^T_{(1)}(\vec{k},t)$ coupled pair. The easiest way to obtain this second set of solutions is to interchange 1 and 2 in the labels and replace i by $-i$ in Equations 6.15 through 6.17.

A simple example serves to show the application of this result. Consider a source free situation with an initial configuration that is a pulse of electric field that is symmetric in a plane. For orientation, chose the plane of symmetry to be the x-y plane. The condition of symmetry implies that the initial field configuration has spatial dependence only in the z coordinate, $\vec{E}(z,t=0)$. Equation 6.9 then requires that

$$\vec{\mathcal{E}}(\vec{k},0) = \sqrt{2\pi}\delta(k_x)\delta(k_y)\int_{-\infty}^{\infty} dz\, e^{-ik_z z}\vec{E}(z,0).$$

Similarly, Maxwell's equations dynamical equations, the transversality condition, and the fact that the $k_x = k_y = 0$ implies that $\vec{\mathcal{E}}(k_z,0)$ and thus also $\vec{E}(z,0)$ cannot be along the z axis. For simplicity, we can take the x axis along the direction of the initial $\vec{E}$ field. Choosing this direction to be the direction of $\hat{\epsilon}^{(1)}$. This then implies the $\hat{e}^{(1)} = \hat{x}$ and $\hat{e}^{(2)} = \hat{y}$. Thus, we have $\mathcal{B}^T_{(2)}(\vec{k},0) = \mathcal{E}^T_{(2)}(\vec{k},0) = \mathcal{B}^T_{(1)}(\vec{k},0) = 0$. To be concrete, we can choose as our pulse shape a gaussian, $\vec{E}(\vec{x},0) = E_0 e^{-\frac{z^2}{2\sigma^2}}\hat{x}$ with width σ. Since the Fourier transform of a gaussian is a gaussian, the full solution in k-pace is

$$\begin{aligned} \vec{\mathcal{E}}(\vec{k},t) &= \mathcal{E}^T_{(1)}(\vec{k},t)\hat{x} = 2\pi E_0\sigma\delta(k_x)\delta(k_y)e^{-\frac{k_z^2\sigma^2}{2}}\cos(|k_z|ct)\hat{x}\\ c\vec{\mathcal{B}}(\vec{k},t) &= c\mathcal{B}^T_{(2)}(\vec{k},t)\hat{y} = i2\pi E_0\sigma\delta(k_x)\delta(k_y)e^{-\frac{k_z^2\sigma^2}{2}}\sin(|k_z|ct)\hat{y}, \end{aligned}$$

where I have used the fact that $\delta(k_x)$ and $\delta(k_y)$ force $k \equiv \sqrt{k_x^2+k_y^2+k_z^2} = |k_z|$.

An important point to note is that the k_x and k_y dependence is completely localized at $k_x = 0$ and $k_y = 0$ whereas the x and y dependence is uniform for all values. This inversion relation also obtains in the k_z case. The non-zero values of the field in k_z is in the range $k_z = \pm\,\text{a few}\,(\frac{1}{\sigma})$ whereas the range of non-zero field in z is $z = \pm\,\text{a few}\,\sigma$. The more spread in configuration space the tighter the field in k-space or, better said, there is an uncertainty like relation between the ranges, $\Delta k_z \times \Delta z \approx 1$.

Fourier transforming back to configuration space the full time dependent solution is

$$\vec{E}(\vec{x},t) = \frac{E_0}{2}\left\{e^{-\frac{(z-ct)^2}{2\sigma^2}} + e^{-\frac{(z+ct)^2}{2\sigma^2}}\right\}\hat{x}.$$

The corresponding $\vec{B}$ field is

$$c\vec{B}(\vec{x},t) = \frac{E_0}{2}\left\{e^{-\frac{(z+ct)^2}{2\sigma^2}} - e^{-\frac{(z-ct)^2}{2\sigma^2}}\right\}\hat{y}.$$

The initial field in $\vec{E}$ produces two travelers, a right and left traveler, in the one dimension available, z, in the same shape as the original pulse with half amplitude.

This solution also has an interesting structure given its genesis. The original, $t = 0$, fields are $\vec{E}(\vec{x},0) = E_0 e^{-\frac{z^2}{2\sigma^2}}\hat{x}$ and $\vec{B}(\vec{x},0) = 0$ which are transverse and agree with our solution. Unstated but assumed is that given the fields at $t = 0$, we want to know the fields at later times, $t \geq 0$. This solution though is valid for all t with the two traveling $\vec{B}(\vec{x},t)$ fields conveniently canceling at $t = 0$. We found this solution by solving the first order in time simple harmonic oscillator differential equation.

How general is this result? The intuition of the string system, Chapter 2.2, tells us that all solutions are travelers formed from the initial configuration of the fields. This is a general result for any shape and exactly the pattern of development as worked in the stretched string, Section 2.2.1, or any $(1,1)$ dimensional hyperbolic system. Here the initial plane symmetric $\vec{E}$ is used to establish the direction of propagation and orientation of the travelers. The associated traveling $\vec{B}$ field is zero in the initial configuration because of cancelation from the overlap of the two oppositely moving travelers. This is a common feature of $(1,1)$ dimensional wave fields even though the relationship of the two fields, $\vec{E}$ and $\vec{B}$, for the electromagnetic field and y and v, for the stretched string, are based on a different dynamic.

We obtained this result because of the fact that we forced the system to be basically a (1,1) system by forcing the plane wave symmetry on the system. Is this technique restricted to situations for which only the plane waves are appropriate? Can any configuration of starting transverse fields be represented and solved?

One beauty of the Maxwell system is the linearity of the theory. In this case, superposition can allow for the construction of more complex cases. In other words, is the most general source free field configuration composed of linear combinations of these plane wave solutions. These are the radiation solutions and they are composed of these plane wave travelers as their basic constituents. In other words, the basic constituents of the classical radiation field, like the photons of quantum field theory, are the plane wave solutions of Equations 6.14. Because of the constraints in Maxwell's equations, all source free fields must be transverse including the initial configurations. These when transformed to $\vec{k}$ are the basis for the travelers that emerge.

In order to understand this idea and make it rigorous, we must construct a consistent complete set of plane wave solutions that spans the space of transverse source free solutions to Maxwell's equations. This will req

Let's work an example. Start with a

If the only system that we had was the source free electromagnetic fields, we can quickly

6.2 Another Representation

The Fourier transformation is not the only set of functions that are complete in a three dimensional space-time. Any of these sets can be used to expand the transverse fields. There is an especially simple set that is a natural for expansion of the electromagnetic fields. This is the set of vector spherical harmonics discussed in Appendix F. This is an expansion that is directly related to the rotational properties of the fields. In addition as discussed in Appendix B.4, for most applications of Fourier transform techniques, the angular integration required for the Fourier transforms in three space treats the unit circle without transformation. This allows these techniques to apply to both the original field configurations and to the Fourier transformations.

The vector spherical harmonic expansions are based on transformations

$$e_{JM(i)}(r,t) \equiv \int d^2\Omega_{\hat{r}} \vec{Y}_{JM}^{(i)^*}(\hat{r}) \cdot \vec{E}\,(\vec{r}.t) \tag{6.18}$$

$$\vec{E}\,(\vec{r}.t) \equiv \sum_{JM(i)} e_{JM(i)}(r,t) \vec{Y}_{JM}^{(i)}(\hat{r}) \tag{6.19}$$

with a similar set of equations for the magnetic field and the $\vec{Y}^{(i)^*}_{JM}(\hat{r})$ are the Hansen vector spherical harmonics, Equation F.90.

In this representation, it is easy to isolate the transverse fields; the sum in Equation 6.19 is restricted to the two cases, $(i) = (e)$ or $(i) = (m)$. One conclusion that is then immediately apparent is that there is no $J = 0$, rotationally unchanged, transverse modes. Since we already know that the transverse fields are the traveling modes, there is no such thing as monopole radiation.

We can find the source free evolution equations for the scalar functions, $e_{JM(i)}(r,t)$.

$$
\begin{aligned}
\frac{\partial e_{JM(e)}(r,t)}{\partial t} &= \int d^2\Omega_{\hat{r}} \vec{Y}^{(e)^*}_{JM}(\hat{r}) \cdot \left(\frac{\partial \vec{E}(\vec{r}.t)}{\partial t} \right) \\
&= c^2 \int d^2\Omega_{\hat{r}} \vec{Y}^{(e)^*}_{JM}(\hat{r}) \cdot \left(\vec{\nabla} \times \vec{B}(\vec{r}.t) \right) \\
&= \sum_{J'M'(i')} c^2 \int d^2\Omega_{\hat{r}} \vec{Y}^{(e)^*}_{JM}(\hat{r}) \cdot \left(\vec{\nabla} \times \left(b_{J'M'(i')}(r,t) \vec{Y}^{(i')}_{J'M'}(\hat{r}) \right) \right) \\
&= \sum_{J'M'(i')} c^2 \int d^2\Omega_{\hat{r}} \vec{Y}^{(e)^*}_{JM}(\hat{r}) \\
&\quad \cdot i \frac{1}{2J'+1} \left(\frac{d}{dr} + \frac{2J'+1}{r} \right) \left(b_{J'M'(i')}(r,t) \vec{Y}^{(i')}_{J'M'}(\hat{r}) \right) \\
&= i \frac{c^2}{2J+1} \left(\frac{d}{dr} + \frac{2J+1}{r} \right) b_{JM(e)}(r,t) \qquad (6.20)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial b_{JM(e)}(r,t)}{\partial t} &= \int d^2\Omega_{\hat{r}} \vec{Y}^{(e)^*}_{JM}(\hat{r}) \cdot \left(\frac{\partial \vec{B}(\vec{r}.t)}{\partial t} \right) \\
&= -i \frac{1}{2J+1} \left(\frac{d}{dr} + \frac{2J+1}{r} \right) e_{JM(e)}(r,t). \qquad (6.21)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\frac{\partial e_{JM(m)}(r,t)}{\partial t} &= \int d^2\Omega_{\hat{r}} \vec{Y}^{(m)^*}_{JM}(\hat{r}) \cdot \left(\frac{\partial \vec{E}(\vec{r}.t)}{\partial t} \right) \\
&= i \frac{c^2}{2J+1} \left(\frac{d}{dr} + \frac{2J+1}{r} \right) b_{JM(m)}(r,t) \qquad (6.22)
\end{aligned}
$$

and

$$\begin{aligned}\frac{\partial b_{JM(m)}(r,t)}{\partial t} &= \int d^2\Omega_{\hat{r}} \vec{Y}^{(m)^*}_{JM}(\hat{r}) \cdot \left(\frac{\partial \vec{B}\,(\vec{r}.t)}{\partial t}\right) \\ &= -i\frac{1}{2J+1}\left(\frac{d}{dr}+\frac{2J+1}{r}\right) e_{JM(m)}(r,t). \end{aligned} \quad (6.23)$$

From these it follows that

$$\frac{\partial 2 e_{JM(i)}(r,t)}{\partial t^2} - \quad (6.24)$$

the $(1,1)$

6.3 Source Free Field Action

In order to produce a mechanical description of these two field systems, we eliminate the $\mathcal{B}$ fields from the dynamical equations, Equations 6.14, which manifests the oscillator nature of the the solutions.

$$\begin{aligned}\frac{1}{c^2}\frac{\partial^2 \mathcal{E}^T_{(1)}}{\partial t^2}(\vec{k},t) &= -k^2 \mathcal{E}^T_{(1)}(\vec{k},t) \\ \frac{1}{c^2}\frac{\partial^2 \mathcal{E}^T_{(2)}}{\partial t^2}(\vec{k},t) &= -k^2 \mathcal{E}^T_{(2)}.(\vec{k},t). \end{aligned} \quad (6.25)$$

In the modern approach to mechanics, an action which reproduces the dynamics is the basis for a full mechanical description. In the case of the source free electromagnetic fields in $\vec{k}$ space each of the two degrees of freedom are simple oscillators and the action is thus well known. In this case though, there is still some ambiguity since the dynamic is linear homogeneous in the fields and there is thus no means to set the scale of the fields $\vec{E}$ and $\vec{B}$, i. e. the units of the field strengths. Consistent with our field theoretic approach, the scale of the fields is set by the dimensional requirements. The dynamic, Equations 6.14, does set the relative dimensions, $\mathcal{B}^T_{(2)}(\vec{k},t) \overset{\text{dim}}{=} \frac{T}{L}\mathcal{E}^T_{(1)}(\vec{k},t)$ where in this equation T is a time dimension and L is a length dimension. Remember that $\vec{k}$ has the dimensions of an inverse length, see Equation 6.9. In addition, the units of the basic source free equations, require that a parameter of dimension $\frac{L}{T}$ be identified for the traveler solutions to make sense. In this regard the c is a required conversion factor that transforms times into lengths.

A "natural" system of units for this case would be to assign a unit to the field such that a quadratic form of the time derivatives of the field be a Lagrangian or a part of one; for example

$$Lag \sim \frac{1}{2}\dot{\vec{\mathcal{E}}}^2$$

for each $\vec{k}$ serving as a kinetic energy like term. On taking the variation of this form with the field to identify the dynamical equations, this construction will yield the required first order differential system. Since the action is

$$S = \int dt Lag,$$

the Lagrangian has dimension

$$Lag \overset{\text{dim}}{=} \frac{ML^2}{T^2}, \tag{6.26}$$

the dimension of a kinetic energy, which implies that the field $\vec{\mathcal{E}}$ has the dimension

$$\vec{\mathcal{E}} \overset{\text{dim}}{=} \left(ML^2\right)^{\frac{1}{2}}. \tag{6.27}$$

Using Equation 6.9, the dimensions of the electric field are then

$$\vec{E} \overset{\text{dim}}{=} \frac{M^{\frac{1}{2}}}{L^2}. \tag{6.28}$$

Diversion on Units

In the usual approach to electromagnetism, the dimension of $\vec{E}$ is given from its definition drawn from Gauss's Law, $\vec{\nabla} \cdot \vec{E}(\vec{x}, t) = \frac{\rho(\vec{x},t)}{\epsilon}$ and the electric part of the Lorentz force equation, $\vec{f}(\vec{x}, t) \equiv \rho(\vec{x}, t)\vec{E}(\vec{x}, t)$. Note that the subscript on the ϵ has been dropped. This will allow an analysis of unit systems beside the rationalized SI system. Assuming that there is an independent unit for charge, Q, these equations imply that $\frac{Q^2}{\epsilon} \overset{\text{dim}}{=} \frac{ML^3}{T^2}$. Actually, the definition of charge is the time integral of a current or $Q \overset{\text{dim}}{=} iT$ or in the rationalized SI system the coulomb is an ampere second. In this form the proper relationship is $\frac{i^2}{\epsilon} \overset{\text{dim}}{=} \frac{ML^3}{T^4}$. This equation makes it clear that the ϵ is a factor that is required to convert the charge units into mechanical units. A similar analysis of the requirement that $\frac{\partial \vec{E}}{\partial t}(\vec{x}, t) = c^2\vec{\nabla} \times \vec{B}(\vec{x}, t) - c^2\mu_0\vec{\text{j}}(\vec{x}, t)$ be dimensionally homogeneous yields

$B \stackrel{\text{dim}}{=} \mu \frac{i}{L}$ and combined with the magnetic portion of the Lorentz force yields $\mu i^2 \stackrel{\text{dim}}{=} \frac{ML}{T^2}$. Again, the μ plays the role of converting the current units to mechanical units. Since we really treat currents only as moving charges, these two conversions must be related. There really is no independent unit for the source in the dynamic of the magnetic field. This restriction of currents to being moving charges relates them dimensionally but also colors our view of the sources for the electromagnetic field and this has a significant impact on the Galilean invariance of our equations. This is discussed in Section 8.2. Solving for i^2, this then constrains the dimensional content of each of these correction factors to satisfy $\frac{1}{\mu\epsilon} \stackrel{\text{dim}}{=} \frac{L^2}{T^2}$. Not only that but since the travelers of the source free system are light, the velocity squared must be the speed of light. It is this identification that allowed us to replace $\frac{1}{\epsilon\mu}$ with c^2 from the beginning in our form of Maxwell's equations. With this understanding of the dimensional content of constants of electromagnetism, we can discuss several options. In the CGS ESU system, ϵ is chosen to be 1 implying that there are no independent units of charge. In this case, the μ attached to the current source must be fit to guarantee that $\frac{1}{\mu}$ is the speed of light in the CGS sytem, $3 \times 10^{10} \frac{\text{cm}}{\text{sec}}$. in the rationalized MKS SI system, the permeability of the vacuum, μ_0 is defined to be $4\pi \times 10^{-7} \frac{N}{A^2}$. The 4π part of the definition is geometric. It is the area of the unit sphere in units in which the area of the unit square is 1. If area had units independently of length units, we would need a conversion factor for areas from L^2. For example areas are in units of scrums. The unit circle is one scrum and the conversion from scrums to the other dimensional description of area, L^2, in which the area of the unit square is 1. For instance, in the MKS system, this implies that 1 $\text{scrum}_{\text{MKS}} = 4\pi\ \text{M}^2$.

Equation 5.6, $\vec{E} \equiv \lim_{Q\to 0} \frac{\vec{F}_Q}{Q}$ where $\vec{F}_Q$ is the force on a small test charge Q. Inserting the usual form of Coulomb's Law, the dimension of $\vec{E} \stackrel{\text{dim}}{=} \frac{ML}{T^2 Q}$. Coulomb's law then requires $\frac{Q^2}{\epsilon} \stackrel{\text{dim}}{=} \frac{ML^3}{T^2}$ where ϵ is a dimension that is set by the unit system for charges being used. In other words, ϵ is a conversion factor for returning charge units to mechanical units. For example, in the electrostatic system of units $\epsilon = 1$ and there is no independent unit of charge. For the rationalized SI units which are the most commonly used units, a separate unit of charge is defined from the the ampere and time, the coulomb C. In this case, the the electric field has dimension $\frac{ML}{T^2 Q}$. This would be an inappropriate approach in a source free setting but if we want to be consistent with the usual definitions we need to connect with the usual approach. Our form of Maxwell's equations, Equations 5.1, are those of the usual SI form and with the two dimensional factors ϵ_0 and μ_0 located on the two source terms only. In the original form of Maxwell's equations, the c^2s in Equations 5.1 were $c^2 \equiv \frac{1}{\epsilon_0\mu_0}$. Although motivated by concerns about local causality, it was Maxwell's great contribution to realize that the displacement current, the term, $\frac{1}{\mu_0}\frac{\partial \vec{E}}{\partial t}(\vec{x},t)$ in Equations 5.1, was required for charge conservation and, once incorporated, allowed in certain cases for the field dynamic to be hyperbolic. The existence of this term in the equations then implies that there are travelers and these have the velocity $\sqrt{\frac{1}{\epsilon_0\mu_0}}$. When he realized this combination of the unit terms had a numeric value close to that of the speed of light, he concluded that light was the traveler solutions of the electromagnetic field dynamic.

This "natural" system of units for the fields could have been arrive at by using the fact that all disturbances of the transverse source free field are travelers we the scale and thus dimensions of $\vec{E}$ can be set by measuring the energy absorbed into a perfectly absorbing boundary. It is interesting to speculate that, had Fresnel realized that the transverse vector amplitude that he was dealing with when he developed his wave theory of light was an electric phenomena, he would have defined a field strength unit that was a result of the energy transmitted when the light was absorbed and given us a unit for the field strength based on mechanical units and that the definition above that comes from the almost contemporaneous work of Faraday in England with the longitudinal part of the field which leads to the definition above may have never been introduced.

6.3.1 Return to the Action for E&M

Regardless of the historical precedents, in order to fully develop the mechanical basis of the source free electromagnetic fields, we will have to identify the action in detail.

From Equations 6.25, there are two oscillators for each $\vec{k}$ and thus the Lagrangian for our source free system is

$$\begin{aligned} L_{SF}(\vec{k},t) &= \sum_{i=1,2}\lambda(\vec{k})\left\{\frac{1}{2}\left(\frac{\partial \mathcal{E}^{T*}_{(i)}}{\partial t}(\vec{k},t)\right)\left(\frac{\partial \mathcal{E}^{T}_{(i)}}{\partial t}(\vec{k},t)\right)\right. \\ &\qquad \left. -\frac{(kc)^2}{2}\left(\mathcal{E}^{T*}_{(i)}(\vec{k},t)\right)\left(\mathcal{E}^{T}_{(i)}(\vec{k},t)\right)\right\}, \qquad (6.29) \end{aligned}$$

where $\lambda(\vec{k})$ is the dimensional factor which brings the Lagrangian into energy units and the variational degrees of freedom which produce the dynamic, Equation 6.25, are $\mathcal{E}^{T}_{(i)}(\vec{k},t)$ and $\mathcal{E}^{T^*}_{(i)}(\vec{k},t)$ and the subscript SF indicates that this is for the source free case. It is important to note that, because of the reality of the $\vec{E}(\vec{x},t)$, in order to not double the degrees of freedom, the variables $\mathcal{E}^{T}_{(i)}(-\vec{k},t) = \mathcal{E}^{T^*}_{(i)}(\vec{k},t)$, see B.2, should not be counted twice; the range of $\vec{k}$ is only over the half sphere.

There are conjugate momenta,

$$\begin{aligned}
p_{(1)}(\vec{k}.t) &\equiv \frac{\partial L(\vec{k})}{\partial \frac{\partial \mathcal{E}^{T}_{(1)}}{\partial t}} = \frac{\lambda(\vec{k})}{2}\frac{\partial \mathcal{E}^{T^*}_{(1)}}{\partial t} = -ikc^2\frac{\lambda(\vec{k})}{2}\mathcal{B}^{T^*}_{(2)}(\vec{k},t) \\
p^*_{(1)}(\vec{k}.t) &\equiv \frac{\partial L(\vec{k})}{\partial \frac{\partial \mathcal{E}^{T^*}_{(1)}}{\partial t}} = \frac{\lambda(\vec{k})}{2}\frac{\partial \mathcal{E}^{T}_{(1)}}{\partial t} = ikc^2\frac{\lambda(\vec{k})}{2}\mathcal{B}^{T}_{(2)}(\vec{k},t) \\
p_{(2)}(\vec{k}.t) &\equiv \frac{\partial L(\vec{k})}{\partial \frac{\partial \mathcal{E}^{T}_{(2)}}{\partial t}} = \frac{\lambda(\vec{k})}{2}\frac{\partial \mathcal{E}^{T^*}_{(2)}}{\partial t} = ikc^2\frac{\lambda(\vec{k})}{2}\mathcal{B}^{T^*}_{(1)}(\vec{k},t) \\
p^*_{(2)}(\vec{k}.t) &\equiv \frac{\partial L(\vec{k})}{\partial \frac{\partial \mathcal{E}^{T^*}_{(2)}}{\partial t}} = \frac{\lambda(\vec{k})}{2}\frac{\partial \mathcal{E}^{T}_{(2)}}{\partial t} = -ikc^2\frac{\lambda(\vec{k})}{2}\mathcal{B}^{T}_{(1)}(\vec{k},t). \qquad (6.30)
\end{aligned}$$

where the last substitution in each line uses Equation 6.14 and, in this sense of Lagrangian dynamics, provides a definition of the magnetic field, $\mathcal{B}^{T}_{(i)}(\vec{k},t)$, and these terms within a multiplicative factor now playing the roll of the conjugate momenta. With these in hand, we can construct the $\vec{k}$ space Hamiltonian for this system:

$$H_{SF}(\vec{k},t) = \sum_{i=1,2}\frac{\lambda(\vec{k})}{2}(kc)^2\left\{\mathcal{E}^{T^*}_{(i)}(\vec{k},t)\mathcal{E}^{T}_{(i)}(\vec{k},t) + c^2\mathcal{B}^{T^*}_{(i)}(\vec{k},t)\mathcal{B}^{T}_{(i)}(\vec{k},t)\right\}. \qquad (6.31)$$

The dynamical equations, Equation 6.14, are the resulting Hamilton's equations. Using the completeness of the $\hat{e}^{(i)}$ for the transverse fields and the fact that all the fields in the source free case are transverse,

$$H_{SF}(\vec{k},t) = \frac{\lambda(\vec{k})}{2}(kc)^2\left\{\vec{\mathcal{E}}^{T^*}(\vec{k},t)\cdot\vec{\mathcal{E}}^{T}(\vec{k},t) + c^2\vec{\mathcal{B}}^{T^*}(\vec{k},t)\cdot\vec{\mathcal{B}}^{T}(\vec{k},t).\right\} \qquad (6.32)$$

Adding in the longitudinal terms which are zero in the source free case to write the full $\vec{k}$ space Hamiltonian,

$$H_{SF}(\vec{k},t) = \frac{\lambda(\vec{k})}{2}(kc)^2\left\{\vec{\mathcal{E}}^{*}(\vec{k},t)\cdot\vec{\mathcal{E}}(\vec{k},t) + c^2\vec{\mathcal{B}}^{*}(\vec{k},t)\cdot\vec{\mathcal{B}}(\vec{k},t)\right\}. \qquad (6.33)$$

Summing over all the degrees of freedom, the total Hamiltonian for this system is

$$H_{SF}(t) = \int_{-\infty}^{\infty} d^3\vec{k}\frac{\lambda(\vec{k})}{4}(kc)^2 \left\{\vec{\mathcal{E}}^*(\vec{k},t)\cdot\vec{\mathcal{E}}(\vec{k},t) + c^2\vec{\mathcal{B}}^*(\vec{k},t)\cdot\vec{\mathcal{B}}(\vec{k},t)\right\} \tag{6.34}$$

This Hamiltonian emerged from a Lagrangian that was time translation invariant and thus there is a conserved quantity, the energy, see Apendix A. Although the Hamiltonian is still expressed in terms of the $\vec{k}$ space description, we can see that the choice $\lambda(\vec{k}) = \frac{\epsilon_0}{(kc)^2}$ will yield an especially simple form. Later, we can show that ϵ_0 is the usual permittivity of the vacuum. Making this substitution and, using the Parseval Identity, Equation B.19, the Hamiltonian becomes

$$H_{SF}(t) = \int_{-\infty}^{\infty} d^3\vec{x}\frac{\epsilon_0}{4}\left\{\vec{E}(\vec{x},t)\cdot\vec{E}(\vec{x},t) + c^2\vec{B}(\vec{x},t)\cdot\vec{B}(\vec{x},t)\right\}. \tag{6.35}$$

Thus we identify the source free energy density, $\rho_{e_{SF}}(\vec{x},t)$, as

$$\rho_{e_{SF}}(\vec{x},t) \equiv \frac{\epsilon_0}{4}\left\{\vec{E}(\vec{x},t)\cdot\vec{E}(\vec{x},t) + c^2\vec{B}(\vec{x},t)\cdot\vec{B}(\vec{x},t)\right\} \tag{6.36}$$

and, from Equation 6.6, the local current density which balances energy density changes is

$$\vec{j}_{E_{SF}}(\vec{x},t) \equiv -\frac{\epsilon_0 c^2}{2}\left(\vec{E}(\vec{x},t)\times\vec{B}(\vec{x},t)\right). \tag{6.37}$$

In a similar fashion, this Hamiltonian and the corresponding action are space translation symmetric.

6.4 A Theory of Photons

6.4.1 Introduction

Although our purpose is a classical description of electromagnetic phenomena, the construction of these solutions of the source free field based on simple harmonic oscillators is a natural and straight forward vehicle for the construction of a quantum field theory of electromagnetic phenomena. If we deal with the source free sector of the theory, we would be developing a theory of photons. The full theory of Quantum Electrodynamics is a theory of electrons and photons.

The route to this formulation is to simply make the oscillators in Section 6.1 quantum oscillators. This is not the usual path to a quantum theory of a field system. In that case, from the classical action of the fields, the mechanical degrees of freedom are identified and the field version of the Poisson bracket is related to the commutation relations appropriate to the a quantum system, see Appendix 10. In our case, we have developed the action for our fields from the identification of the intrinsic oscillator nature of the source free fields. It should be clear that these two routes to quantization lead to the same construction of a quantum system. The route used here has the advantage of interpretational simplicity and much of the constructional ambiguities that emerge in the direct field theory approach are managed by the power of the Fourier transform machinery. In the following subsection we review the properties of the simple quantum mechanical oscillator, particular the relationship to the classical oscillator. Following that construction, we develop our theoretical construction of the photons. These are simple photons. There are no charged particles and construction of the full quantum field theory requires the presence of the charges. This will require the full machinery of quantum field theory and will be saved for later development, see Section **??**.

6.4.2 The Quantum Classical Oscillator

The quantum oscillator is defined by the Hamiltonian

$$H = \frac{p^2}{2m} + k\frac{q^2}{2} \tag{6.38}$$

and the commutation relationships for the hermitian operators, q and p; $[q,q] = [p,p] = 0$ and $[q,p] = i\hbar$.

Defining the the non-Hermitean operator $a = \sqrt{\frac{m\omega}{2\hbar}}q + i\frac{p}{\sqrt{2m\omega\hbar}}$ where $\omega \equiv \sqrt{\frac{k}{m}}$. It is important to note that each term of a is dimensionless and ω is dimensionally a frequency but we have no reason for interpreting it as an oscillator frequency. Granted, when Equation 6.38 is a classical Hamiltonian, ω is the radian frequency of the oscillator.

The commutation relations for the operator a and its adjoint become $[a,a] = [a^\dagger, a^\dagger] = 0$ and $[a, a^\dagger] = 1$. It follows that $[a, a^\dagger a] = a$ and $[a^\dagger, a^\dagger a] = -a^\dagger$. Thus, the spectrum is a ladder spectrum with a moving down the ladder and $a^\dagger$ moving up where, since the Hamiltonian in Equation 6.38 which is now expressed as

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \tag{6.39}$$

is positive definite, there is a lowest step on the ladder,

$$a|0>= 0. \tag{6.40}$$

Thus the ladder states generated by

$$|n>\equiv \frac{a^{\dagger n}}{\sqrt{n!}}|0>, n = 0, 1, 2, \cdots \tag{6.41}$$

are the stationary states with energy $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$. The states $|n>$ form an orthonormal basis, $<n|m>= \delta_{nm}$, are complete, $\sum_{n=0}^{\infty} |n><n| = \mathbf{1}$, and are the eigenstates of the operator $a^\dagger a$ with eigenvalue n.

Using Equation 6.40, we can find the representation of these stationary states in the coordinate or q representation. The ground state satisfies

$$0 = \int dq' <q|a|q'><q'|0>= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{m\omega}{\hbar}q + \frac{\partial}{\partial q}\right)\psi_0(q) \tag{6.42}$$

whose solution is the well known Gaussian with a standard deviation of $\sqrt{\frac{\hbar}{m\omega}}$ and which is normalized to $\int_{-\infty}^{\infty}\psi_0^2(q)dq = 1$,

$$\psi_0(q) = \sqrt[4]{\frac{m\omega}{\hbar\pi}}e^{-\frac{\frac{m\omega}{\hbar}q^2}{2}}, \tag{6.43}$$

and where $\psi_0(q) =<q|0>$. The probability distribution in q, $\psi_0^*(q)\psi_0(q)$, has mean of zero and a standard deviation of $\sqrt{\frac{\hbar}{2m\omega}}$. Since the expectation value for the position in the ground state,

$$_0< q >_0= 0, \tag{6.44}$$

where the label 0 on the brackets indicates the state used for the evaluation of the expectation value and, since the variance[2] of a Gaussian is the square of the standard deviation,

$$_0<< q >>_0=_0< q^2 >_0= \frac{\hbar}{2m\omega}. \tag{6.45}$$

All the higher ladder states wave functions can be found from Equation 6.41,

$$<q|n>\equiv \psi_n(q) = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\hbar}q - \frac{\partial}{\partial q}\right)^n\left(\frac{\hbar}{2m\omega}\right)^{\frac{n}{2}}\psi_0(q). \tag{6.46}$$

[2]The variance of an operator A in the state $|\beta>$ is defined as $_\beta<< A >>_\beta=_\beta< A^2 >_\beta - (_\beta< A >_\beta)^2$. Note that $_\beta<< A >>_\beta$ has the dimensional content of A^2.

These are the generator equations for the Hermite functions.

Another important state for our purposes is the state $|\alpha>$ which is the eigenstate of the ladder operator a or $a|\alpha>= \alpha|\alpha>$ where α is an arbitrary complex number[3]. This state is generated by the displacement operator

$$D(\alpha) \equiv e^{\left(\alpha a^\dagger - \alpha^* a\right)} \tag{6.47}$$

operating on the ground state,

$$|\alpha>= e^{\left(\alpha a^\dagger - \alpha^* a\right)}|0> . \tag{6.48}$$

These states are normalized and are expanded in the ladder basis using the Baker-Campbell-Hausdorf expansion[4] as

$$\begin{aligned} |\alpha> &= e^{\left(\alpha a^\dagger\right)} e^{-(\alpha^* a)} e^{-\frac{|\alpha|^2}{2}[a.a^\dagger]}|0> \\ &= e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{n=\infty} \frac{\alpha^n}{\sqrt{n!}}|n> . \end{aligned} \tag{6.49}$$

That this state is an eigenstate of the operator a with eigenvalue α can be verified from the operator equations, $D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha)$ and $D^\dagger(\alpha) a D(\alpha) = a + \alpha$. These states are called coherent states. The coherent states are an over-complete set and the completeness relation is

$$\frac{1}{\pi}\int |\alpha><\alpha| d^2\alpha = 1. \tag{6.50}$$

Another important property of the coherent states is that the probability of occupation of the state $|n>$ is

$$P(n,\alpha) \equiv |<n|\alpha>|^2 = \frac{e^{-|\alpha|^2}\left(|\alpha|^2\right)^n}{n!} \tag{6.51}$$

which is a Poisson distribution with mean $|\alpha|^2$ and standard deviation $|\alpha|$.

Two special cases will be especially important to us, α real and α imaginary. For the case $\alpha = d$ real, the operator $D(d) = e^{-i\sqrt{\frac{2}{\omega\hbar m}}pd}$ in the q representation becomes

$$D_q(d) = e^{-\sqrt{\frac{2\hbar}{\omega m}}d\frac{\partial}{\partial q}}. \tag{6.52}$$

[3]Since the operator a is not Hermitian its eigenfunctions have complex eigenvalues.

[4]As a special case, $e^{A+B} = e^A e^B e^{-[A,B]\frac{1}{2}}$, if $[A,[A,B]] = [B,[A,B]] = 0$.

Since for any smooth function of q

$$e^{\gamma\frac{\partial}{\partial q}}f(q)=\sum_{n=0}^{\infty}\frac{\gamma^n}{n!}\frac{\partial^n}{\partial q^n}f(q)=f(q+\gamma),$$

the operator $D_q(d)$ translates any wave function in the q representation by the amount $-\sqrt{\frac{2\hbar}{\omega m}}d$. Similarly, if α is imaginary, $\alpha = id$, $D(id) = e^{id\sqrt{\frac{2m\omega}{\hbar}}q}$. In the p representation, q is $i\hbar\frac{\partial}{\partial p}$ and thus

$$D_p(id) = e^{-\sqrt{2m\omega\hbar}d\frac{\partial}{\partial p}} \tag{6.53}$$

which translates any wave function in the p representation by the amount $-\sqrt{2\hbar\omega m}d$.

Applying Equation 6.52 to the ground state in the q representation, Equation 6.43,

$$\psi_{\sqrt{\frac{2\hbar}{\omega m}}d}(q) \equiv e^{-\sqrt{\frac{2\hbar}{m\omega}}d\frac{\partial}{\partial q}}\psi_0(q) = \sqrt[4]{\frac{m\omega}{\hbar\pi}}e^{-\frac{\frac{m\omega}{\hbar}\left(q-\sqrt{\frac{2\hbar}{m\omega}}d\right)^2}{2}}, \tag{6.54}$$

we now have a state that has minimum uncertainty and is a Gaussian with standard deviation $\sqrt{\frac{\hbar}{m\omega}}$ and mean

$$q_d \equiv \sqrt{\frac{2\hbar}{m\omega}}d. \tag{6.55}$$

The probability distribution of this state is also a Gaussian with the same mean but now with standard deviation $\sqrt{\frac{\hbar}{2m\omega}}$.

This new state, Equation 6.54, is not stationary. The evolution is best understood in terms of the the evolution of the underlying ladder states. Although the ladder states are stationary, their probability distributions are time independent, their wave functions have a time dependence in the phase and, since each ladder state has a different phase development, superpositions of these states do have an evolution through interference[5]. In other words, the time dependence of the displaced state can be found by adding the time dependence to Equation 6.49,

$$\psi_{q_d}(q,t) \quad = \quad e^{-\frac{d^2}{2}}\sum_{n=0}^{\infty}\frac{d^n}{\sqrt{n!}}\psi_n(q,t)$$

[5]Alternatively, $\psi_{q_d}(q,t) = e^{-i\frac{H}{\hbar}t}\psi_{q_d}(q,0)$ which leads to Equation 6.56 since $H\psi_n(q) = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + m\omega^2\frac{q^2}{2})\psi_n(q) = \hbar\omega\left(n+\frac{1}{2}\right)\psi_n(q)$

$$= e^{-\frac{d^2}{2}} \sum_{n=0}^{\infty} \frac{d^n}{\sqrt{n!}} e^{-i\omega\left(n+\frac{1}{2}\right)t} \psi_n(q) \tag{6.56}$$

$$= e^{-\frac{i\omega t}{2}} e^{-\frac{|de^{-i\omega t}|^2}{2}} \sum_{n=0}^{\infty} \frac{\left(de^{-i\omega t}\right)^n}{\sqrt{n!}} \psi_n(q)$$

$$= e^{-\frac{i\omega t}{2}} D_q(de^{-i\omega t}) \psi_0(q), \tag{6.57}$$

The operator

$$D(de^{-i\omega t}) = e^{\left\{de^{-i\omega t}a^\dagger - de^{i\omega t}a\right\}}$$

$$= e^{-i\left\{\sqrt{\frac{2m\omega}{\hbar}}qd\sin(\omega t) + \sqrt{\frac{2}{m\omega\hbar}}pd\cos(\omega t)\right\}} \tag{6.58}$$

$$= e^{-i\sqrt{\frac{2m\omega}{\hbar}}qd\sin(\omega t)} e^{-i\sqrt{\frac{2}{m\omega\hbar}}pd\cos(\omega t)} e^{\frac{1}{\hbar}d^2\sin(\omega t)\cos(\omega t)[q,p]}, \tag{6.59}$$

where Equation 6.59 follows from Equation 6.58 using the Baker-Campbell-Hausdorf expansion. In the q representation Equation 6.59 becomes

$$D_q(de^{-i\omega t}) = e^{-i\sqrt{\frac{2m\omega}{\hbar}}qd\sin(\omega t)} e^{-\sqrt{\frac{2\hbar}{m\omega}}d\cos(\omega t)\frac{\partial}{\partial q}} e^{id^2\sin(\omega t)\cos(\omega t)}$$

$$= e^{-id\sin(\omega t)\left\{\sqrt{\frac{2m\omega}{\hbar}}q - d\cos(\omega t)\right\}} e^{-q_d\cos(\omega t)\frac{\partial}{\partial q}} \tag{6.60}$$

and thus our evolving wave function is

$$\psi_{q_d}(q,t) = \sqrt[4]{\frac{m\omega}{\hbar\pi}} e^{-i\left\{\frac{\omega t}{2} + d\sin(\omega t)\left\{\sqrt{\frac{2m\omega}{\hbar}}q - d\cos(\omega t)\right\}\right\}} e^{-\frac{\frac{m\omega}{\hbar}\left(q - q_d\cos(\omega t)\right)^2}{2}}. \tag{6.61}$$

The magnitude of Equation 6.61 is

$$\psi^*_{q_d}(q,t)\psi_{q_d}(q,t) = \sqrt{\frac{m\omega}{\hbar\pi}} e^{-\frac{m\omega}{\hbar}(q - q_d\cos\omega t)^2} \tag{6.62}$$

and thus the probability distribution is again a Gaussian but now with mean $q_d \cos\omega t$ and the same standard deviation as the original ground state, $\sqrt{\frac{\hbar}{2m\omega}}$. Also since the distribution in q is a Gaussian, the variance is the standard deviation squared,

$$_{q_d}<< q >>_{q_d} = _0<< q >>_0 = \frac{\hbar}{2m\omega} \tag{6.63}$$

The interpretation of this result is clear. The minimum uncertainty wave packet originally at a distance $q_d \equiv \sqrt{\frac{2\hbar}{\omega m}}d$ from the origin moves without changing shape in the same way a classical oscillator with radian frequency ω would. It is also important to note that, although this result may appear to be a classical limit to the quantum oscillator, it is an exact quantum result; there were no large n approximation or $d \gg 1$ or $d \ll 1$ conditions. It is true that any macroscopic classical system has its displacement from the origin large compared with the variance of the the quantum ground state and thus operationally $q_d \gg \sqrt{\frac{\hbar}{2m\omega}}$ which implies that $d \gg \frac{1}{2}$.

It is worthwhile to develop other features of this special state. The expectation value for the energy which is independent of time is

$$\begin{aligned} {}_{q_d}< E >_{q_d} &= \int_{-\infty}^{\infty} \psi_{q_d}^*(q,t) H \psi_{q_d}(q,t) dq && (6.64) \\ &= \int_{-\infty}^{\infty} \psi_{q_d}^*(q,0) \left(-\frac{1}{2m} \frac{\partial^2}{\partial q^2} + \frac{m\omega^2}{2} q^2 \right) \psi_{q_d}(q,0) dq \\ &= \int_{-\infty}^{\infty} \psi_0^*(q,0) \left(-\frac{1}{2m} \frac{\partial^2}{\partial q^2} + \frac{m\omega^2}{2} (q - q_d)^2 \right) \psi_0(q,0) dq \\ &= {}_0\!< \left(H - m\omega^2 q q_d + \frac{m\omega^2}{2} q_d^2 \right) >_0 \\ &= \frac{1}{2}\hbar\omega + \frac{m\omega^2}{2} q_d^2 && (6.65) \end{aligned}$$

where q_d is the is displacement of the oscillator zero point. As expected this result is the zero point energy plus the classical energy. Similarly, the expectation value of the energy squared is

$$\begin{aligned} {}_{q_d}< E^2 >_{q_d} &= \int_{-\infty}^{\infty} \psi_{q_d}^*(q,t) H^2 \psi_{q_d}(q,t) dq \\ &= {}_0\!< \left(H - m\omega^2 q q_d + \frac{m\omega^2}{2} q_d^2 \right)^2 >_0 \\ &= \left(\frac{1}{2}\hbar\omega + \frac{m\omega^2}{2} q_d^2 \right)^2 + \left(m\omega^2 \right)^2 {}_0\!< q^2 >_0 q_d^2 && (6.66) \end{aligned}$$

or, although the expectation value of the energy is the classical value plus the zero point energy, the variance of the energy is

$$ {}_{q_d}\!<< E >>_{q_d} = \left(m\omega^2 \right)^2 q_d^2 \, {}_0\!<< q >>_0 = \left(m\omega^2 \right)^2 q_d^2 \frac{\hbar}{2m\omega}, \qquad (6.67)$$

and is purely a quantum effect, proportional to $\hbar$ which vanishes in the limit $\hbar \to 0$.

It is important to understand that this oscillator is not the most general case of classical motion possible. Since the classical oscillator equation is second order in time, there is both a starting position and an initial velocity possible. It should be obvious that we can obtain the case of the initial velocity distribution with similar results to the above by doing all of the analysis in the p-representation of the oscillator quantum mechanics. In the p representation, the ground state condition, Equation 6.40, is

$$i\sqrt{\frac{m\omega\hbar}{2}}\left(\frac{\partial}{\partial p}+\frac{p}{m\omega\hbar}\right)\psi_0\left(p\right)=0, \tag{6.68}$$

and the ground state wave function[6] which leads to a normalized probability distribution is

$$\psi_0\left(p\right)=\frac{1}{\sqrt[4]{m\omega\hbar\pi}}e^{-\frac{p^2}{2m\omega\hbar}}, \tag{6.69}$$

which has a mean of 0 and standard deviation of $\sqrt{m\omega\hbar}$. Again, the associated momentum probability distribution has a mean of 0 and standard deviation of $\sqrt{\frac{m\omega\hbar}{2}}$.

Following the same development as before except in the p representation, but using Equation 6.53 or 6.58 with $t=-\frac{\pi}{2\omega}$,

$$\psi_{p_d}\left(p\right)=\frac{1}{\sqrt[4]{\pi m\omega\hbar}}e^{-\frac{\left(p-p_d\right)^2}{2m\omega\hbar}} \tag{6.70}$$

where $p_d\equiv\sqrt{2m\omega\hbar}\,d$ which is a Gaussian with mean of p_d and standard deviation of $\sqrt{m\omega\hbar}$. As before the probability distribution is a Gaussian with mean p_d and standard deviation $\sqrt{\frac{m\omega\hbar}{2}}$.

Using the appropriately weighted Fourier transform, the q representation wave function is

$$\begin{aligned}\psi_{p_d}\left(q\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\frac{pq}{\hbar}}\left\{\frac{1}{\sqrt[4]{\pi m\omega\hbar}}e^{-\frac{\left(p-p_d\right)^2}{2m\omega\hbar}}\right\}\frac{dp}{\sqrt{\hbar}} \\ &= \sqrt[4]{\frac{m\omega}{\pi\hbar}}e^{i\frac{p_dq}{\hbar}}e^{-\frac{m\omega q^2}{2\hbar}}. \end{aligned} \tag{6.71}$$

[6] Another approach to finding the wave function in the p representation would be the use of Fourier transforms, Appendix B, but some care must be exercised since, although p and q are conjugate variables in Lagrange mechanics, the appropriate variables for Fourier transforms are dimensionally inverses of each other, Equation B.29. For example, if q is a length, p is a linear momentum in mechanics but in the Fourier transform case, the appropriate variable is the wave number, $k\equiv\frac{p}{\hbar}$, which is an inverse length. Also the transforms are dimensional inverses but the modulus of the wave function squared is a probability distribution and, thus, the wave function is normed to have dimension that is the inverse square root of the distribution variable.

This is a Gaussian with mean 0 and standard deviation $\sqrt{\frac{\hbar}{m\omega}}$ multiplied by a phase factor and normed so that the modulus squared is normed for a distribution in q.

The temporal evolution follows the same pattern as before.

$$\begin{aligned} \psi_{p_d}(p,t) &= e^{-\frac{i\omega t}{2}} D_p\left(de^{-i\left(\omega t-\frac{\pi}{2}\right)}\right)\psi_0(p) \\ &= \frac{1}{\sqrt[4]{m\omega\hbar\pi}} e^{-\frac{i\omega t}{2}} e^{-id\sin\omega t\left(\sqrt{\frac{2}{m\omega\hbar}}\,p-d\cos\omega t\right)} e^{-\frac{(p-p_d\cos\omega t)^2}{2m\omega\hbar}}. \end{aligned} \tag{6.72}$$

Again, a Gaussian with an extra phase term. The probability distribution changes momentum in the same pattern as a classical oscillator with radian frequency ω.

The evolving q representation wave function is found using the Fourier transformation of Equation 6.72 and is

$$\psi_{p_d}(q,t) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} e^{-i\frac{\omega t}{2}} e^{ip_d\cos\omega t\frac{q}{\hbar}} e^{-\frac{m\omega}{2\hbar}\left(q-\frac{p_d}{m\omega}\sin\omega t\right)^2}. \tag{6.73}$$

This wave function produces a probability distribution in q that is a Gaussian which has a standard deviation of $\sqrt{\frac{\hbar}{2m\omega}}$ and a mean of $\frac{p_d}{m\omega}\sin\omega t$. It moves in space in the same way a classical oscillator with radian frequency ω excited by an initial velocity of $\frac{p_d}{m}$.

Returning to the p representation, we find the expectation of the energy to be ${}_{p_d}< E >_{p_d} = \frac{1}{2}\omega\hbar + \frac{p_d^2}{2m}$, the zero point energy plus the classical energy. The variance follows similarly to the development of Equation 6.67 and is

$$_{p_d}<< E >>_{p_d} = \frac{1}{m^2} p_d^2 \,{}_0<< p >>_0 = \frac{p_d^2}{m^2}\frac{m\omega\hbar}{2}. \tag{6.74}$$

We are now in a position to incorporate the most general initial conditions and allowing it to evolve. Consider the case of a displacement in both the momentum and position of the ground state,

$$\alpha = d_1 + id_2 \equiv De^{i\gamma} \tag{6.75}$$

with $D \equiv \sqrt{d_1^2 + d_2^2}$ and $\gamma \equiv \tan^{-1}\left(\frac{d_2}{d_1}\right)$. The position displacement is $q_{d_1} \equiv \sqrt{\frac{2\hbar}{\omega m}} d_1 = \sqrt{\frac{2\hbar}{\omega m}} D\cos\gamma$ and the momentum displacement is $p_{d_2} \equiv \sqrt{2m\omega\hbar}\, d_2 = \sqrt{2m\omega\hbar}\, D\sin\gamma$.

The analysis of Equations 6.56 through 6.77 holds and leads to the replacement of ωt by $\omega t - \gamma$ in Equation 6.60 to yield,

$$D_q\left(De^{-i(\omega t-\gamma)}\right) = e^{-iD\sin(\omega t-\gamma)\left\{\sqrt{\frac{2m\omega}{\hbar}}q-D\cos(\omega t-\gamma)\right\}}e^{-\sqrt{\frac{2\hbar}{\omega m}}D\cos(\omega t-\gamma)\frac{\partial}{\partial q}}. \tag{6.76}$$

The evolving wave function is

$$\psi_{q_{d_1}p_{d_2}}(q,t) = \sqrt[4]{\frac{m\omega}{\hbar\pi}}e^{-i\left\{\frac{\omega t}{2}+D\sin(\omega t-\gamma)\left\{\sqrt{\frac{2m\omega}{\hbar}}q-D\cos(\omega t-\gamma)\right\}\right\}} \times e^{-\frac{\frac{m\omega}{\hbar}\left(q-q_{d_1}\cos(\omega t)-\frac{p_{d_2}}{m\omega}\sin(\omega t)\right)^2}{2}}. \tag{6.77}$$

As expected, the magnitude is a Gaussian with mean $q_{d_1}\cos(\omega t)+\frac{p_{d_2}}{m\omega}\sin(\omega t)$ and standard deviation $\sqrt{\frac{\hbar}{m\omega}}$. The probability distribution for this state is the magnitude squared and thus has the same mean and the standard deviation is $\sqrt{\frac{\hbar}{2m\omega}}$.

To study the energy expectation and the variance it is easier to work in the operator form of Equation 6.59 with the appropriate argument. Again using the time independence of the expectation value of the energy and following the development of Equation 6.64 through 6.65,

$$\begin{aligned}
{}_{q_{d_1}p_{d_2}}< E >_{q_{d_1}p_{d_2}} &= {}_0< e^{i\sqrt{\frac{2\hbar}{m\omega}}pD\cos\gamma}e^{-i\sqrt{\frac{2m\omega}{\hbar}}qD\sin\gamma}e^{iD^2\cos\gamma\sin\gamma} \\
&\quad\times\left(\frac{p^2}{2m}+\frac{m\omega^2}{2}q^2\right)e^{i\sqrt{\frac{2m\omega}{\hbar}}qD\sin\gamma} \\
&\quad\times e^{-i\sqrt{\frac{2\hbar}{m\omega}}pD\cos\gamma}e^{-iD^2\cos\gamma\sin\gamma} >_0 \\
&= {}_0<\left(e^{-i\sqrt{\frac{2m\omega}{\hbar}}qD\sin\gamma}\frac{p^2}{2m}e^{i\sqrt{\frac{2m\omega}{\hbar}}qD\sin\gamma}\right. \\
&\quad\left.+\frac{m\omega^2}{2}q^2\right) >_0 \\
&= {}_0<\frac{(p-p_{d_2})^2}{2m}+\frac{m\omega^2}{2}(q-q_{d_1})^2 >_0 \\
&= {}_0< H-\frac{pp_{d_2}}{m}+\frac{p_{d_2}^2}{2m}-m\omega^2qq_{d_1}+\frac{m\omega^2}{2}q_{d_1}^2 >_0 \\
&= \frac{\hbar\omega}{2}+\frac{p_{d_2}^2}{2m}+\frac{m\omega^2}{2}q_{d_1}^2
\end{aligned} \tag{6.78}$$

Again, this is the zero point energy plus the classical energy.

Proceeding similarly to find the variance in the energy for this case, we follow the same pattern as Equation 6.66. By putting all the quadratic dependence on the operators p and q in H, the expectation for the energy squared becomes

$$\begin{aligned} {}_{q_{d_1}p_{d_2}}< E^2 >_{q_{d_1}p_{d_2}} &= \left(\frac{1}{2}\hbar\omega + \frac{m\omega^2}{2}q_{d_1}^2 + \frac{p_{d_2}^2}{2m}\right)^2 + \left(m\omega^2\right)^2 {}_0< q^2 >_0 q_{d_2}^2 \\ &\quad + \frac{{}_0< p^2 >_0 p_{d_2}^2}{m^2} + {}_0< \{q,p\}_+ >_0 \omega^2 q_{d_1} p_{d_2} \qquad (6.79) \end{aligned}$$

Where $\{q,p\}_+$ is the anticommutator of q and p and is the only new feature of the energy squared expectation. It is straight forward to show that the vacuum expectation value of this operator is zero[7]. Thus the variance of the energy is the sum of the variances of the separate p and q translates of the ground state. As in the previous cases, this is a purely quantum effect and would vanish in the limit $\hbar$ goes to zero.

6.4.3 The Quantized Electromagnetic Field

This result will be helpful in interpreting the large amplitude fields that are the basis of classical electromagnetism, see Section 6.1 and Equation 6.25; for each $\vec{k}$, these fields are oscillators with radian frequency kc. In a sense, we are now reversing the development of Section 6.3 where we took the mechanical oscillator nature of the source free fields to discover the appropriate action. Following the route to quantum mechanics of using a classical action to produce a quantum theory.

[7]The operator $qp = -i\frac{\hbar}{2}(a+a^\dagger)(a-a^\dagger) = -i\frac{\hbar}{2}(aa - a^\dagger a^\dagger - 1)$ has the vacuum expectation ${}_0< qp >_0 = i\frac{\hbar}{2}$ and pq is the adjoint of qp.