Chapter 7

The Electromagnetic Field with Sources

Reverting back to the full set of Maxwell's equations, Equation 5.1, we see that the magnetic field remains transverse even in the presence of sources. The electric field on the other hand now has a longitudinal component.

Although we could obtain the transverse and longitudinal structure of $\vec{E}$ in configuration space, it is easier to interpret when these equations are written into $\vec{k}$ space. Later in this section, we will return to configuration space. The $\vec{k}$ form of Maxwell's equations and charge conservation, Equations 5.1 and 5.3, are

$$\begin{aligned}
\frac{\partial \vec{\mathcal{E}}}{\partial t}(\vec{k},t) &= ic^2\vec{k}\times\vec{\mathcal{B}}(\vec{k},t) - c^2\mu_0\vec{\jmath}(\vec{k},t) \\
\frac{\partial \vec{\mathcal{B}}}{\partial t}(\vec{k},t) &= -i\vec{k}\times\vec{\mathcal{E}}(\vec{k},t) \\
i\vec{k}\cdot\vec{\mathcal{E}}(\vec{k},t) &= \frac{\wp(\vec{k},t)}{\epsilon_0} \\
\vec{k}\cdot\vec{\mathcal{B}}(\vec{k},t) &= 0.
\end{aligned} \tag{7.1}$$

where $\wp(\vec{k},t)$ is the Fourier transform of the charge distribution and $\vec{\jmath}(\vec{k},t)$ is the transform of the current density. The current conservation equation is

$$\frac{\partial \wp}{\partial t}(\vec{k},t) = i\vec{k}\cdot\vec{\jmath}(\vec{k},t) = i\vec{k}\cdot\vec{j}^{\,\mathrm{L}}(\vec{k},t), \tag{7.2}$$

since by definition $i\vec{k}\cdot\vec{j}^{\,\mathrm{T}}(\vec{k},t) = 0$ and the longitudinal and transverse fields are complete in the sense that $\vec{\jmath}(\vec{k},t) = \vec{j}^{\,\mathrm{T}}(\vec{k},t) + \vec{j}^{\,\mathrm{L}}(\vec{k},t)$. Using the longitudinal projection operator $\vec{j}^{\,\mathrm{L}}(\vec{k},t) = \frac{\vec{k}}{k^2}\left(\vec{k}\cdot\vec{j}(\vec{k},t)\right)$.

Similarly, using the longitudinal projection operator on the first of the Maxwell's equations, the longitudinal electric field satisfies

$$\frac{\partial \vec{\mathcal{E}}^L}{\partial t}(\vec{k},t) = -c^2\mu_0\vec{j}^{\,L}(\vec{k},t) = \frac{1}{\epsilon_0}\frac{\vec{k}}{k^2}\left(\vec{k}\cdot\vec{j}(\vec{k},t)\right) = \frac{-i}{\epsilon_0}\frac{\vec{k}}{k^2}\frac{\partial\wp}{\partial t}(\vec{k},t) \tag{7.3}$$

or the field combination $\vec{\mathcal{E}}^L(\vec{k},t) - \frac{-i}{\epsilon_0}\frac{\vec{k}}{k^2}\wp(\vec{k},t)$ is a constant. This result is consistent with the Maxwell constraint equation $i\vec{k}\cdot\vec{\mathcal{E}}(\vec{k},t) = \frac{\wp(\vec{k},t)}{\epsilon_0}$. Integrating Equation 7.3,

$$\vec{\mathcal{E}}^L(\vec{k},t) - \vec{\mathcal{E}}^L(\vec{k},t_0) = \frac{-i}{\epsilon_0}\frac{\vec{k}}{k^2}\wp(\vec{k},t) - \frac{-i}{\epsilon_0}\frac{\vec{k}}{k^2}\wp(\vec{k},t_0). \tag{7.4}$$

Using the fact that the product of Fourier transforms is the convolution of the configuration space functions, Equation B.20 and Equation B.53, we can find the longitudinal electric field in configuration space,

$$\begin{aligned}\vec{E}^L(\vec{x},t) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} d^3\vec{x}'\,\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\rho(\vec{x}',t) \\ &\quad +\vec{E}^L(\vec{x},t_0) - \frac{1}{4\pi}\int_{-\infty}^{\infty} d^3\vec{x}'\,\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\rho(\vec{x}',t_0).\end{aligned} \tag{7.5}$$

Thus,

$$\vec{E}^L(\vec{x},t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} d^3\vec{x}'\,\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\rho(\vec{x}',t), \tag{7.6}$$

only if

$$\vec{E}^L(\vec{x},t_0) = \frac{1}{4\pi}\int_{-\infty}^{\infty} d^3\vec{x}'\,\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\rho(\vec{x}',t_0). \tag{7.7}$$

This requires that when charges are carried into interaction they must include their associated longitudinal electric fields. The concept of the bare electron without its associated longitudinal field is inconsistent with Maxwell's equations.

The development of these equations is easy to interpret in wave-number space but, since the Fourier Transform is non-local in configuration space, the inverse Fourier transform makes the form of these projection operators in configuration space non-trivial, see Equation 7.5. The transform of the longitudinal projector, $\frac{k_i k_j}{k^2}$, is written in a symbolic form $-\frac{\vec{\nabla}_i\vec{\nabla}_j}{\vec{\nabla}^2}$ and we can obtain the result Equation 7.3 in configuration space directly from Maxwell's Equations, Equation 5.1, and charge conservation, Equation 5.3.

$$\begin{aligned}
\frac{\partial \vec{E}_i^L}{\partial t}(\vec{x},t) &= \sum_j \frac{\vec{\nabla}_i\vec{\nabla}_j}{\vec{\nabla}^2}\left\{c^2\left(\vec{\nabla}\times\vec{B}\right)_j - c^2\mu_0\vec{j}_j\right\} \\
&= -c^2\mu_0\frac{\vec{\nabla}_i}{\vec{\nabla}^2}\left(\vec{\nabla}\cdot\vec{j}\right) \\
&= \frac{\vec{\nabla}_i}{\vec{\nabla}^2}\left(\frac{1}{\epsilon_0}\frac{\partial\rho}{\partial t}\right)
\end{aligned}$$

or

$$\frac{\partial}{\partial t}\left(\vec{E}^L(\vec{x},t) - \frac{1}{\epsilon_0}\frac{\vec{\nabla}}{\vec{\nabla}^2}\rho(\vec{x},t)\right) = 0. \tag{7.8}$$

The difficulty of interpreting this is the $\vec{\nabla}^2$ in the denominator. Again this can be symbolically removed by the realization that a scalar field $\phi(\vec{x},t)$ satisfying

$$\vec{\nabla}^2\phi(\vec{x},t) = \frac{1}{\epsilon_0}\rho(\vec{x},t) \tag{7.9}$$

would provide solutions for $\vec{E}^L(\vec{x},t)$ in the form

$$\frac{\partial}{\partial t}\left(\vec{E}^L(\vec{x},t) - \vec{\nabla}\phi(\vec{x},t)\right) = 0. \tag{7.10}$$

or

$$\vec{E}^L(\vec{x},t) = \vec{\nabla}\phi(\vec{x},t). \tag{7.11}$$

Equation 7.11 along with Equation 7.9 determines the longitudinal electric field for all times. This reduces the problem of finding this part of the field to finding the solution to Equation 7.9. This is the famous Poisson equation for the electromagnetic potential. I have avoided the use of potentials and, in the case of the unbounded field in the source free case, the potential is really not necessary or particularly helpful.

The general technique for solution to the potential problem which is also applicable to the boundless and bounded case is the use of Green's functions, Appendix C. What this technique does is replace finding the solution to a partial differential equation with specified boundary conditions to finding an integral kernel that provides a solution by integration.

For our case of the unbounded field, we can still use the Fourier techniques which imply vanishing behavior at large distances and use them to identify the Green's function appropriate to that case. Fourier transforming Equation 7.9, dividing through by $\vec{k}^2$, and transforming back

$$\phi\left(\vec{x},t\right) = \int d^3\vec{x}' \frac{1}{4\pi\left|\vec{x}-\vec{x}'\right|}\frac{\rho\left(\vec{x}',t\right)}{\epsilon_0} = \int d^3\vec{x}' G\left(\vec{x}|\vec{x}'\right)\frac{\rho\left(\vec{x}',t\right)}{\epsilon_0} \tag{7.12}$$

Yielding the Green's function, $G\left(\vec{x}|\vec{x}'\right) = \frac{1}{4\pi|\vec{x}-\vec{x}'|}$ for the unbounded three space case.

Equation 7.9 is

$$\int \frac{d^3\vec{k}}{(\sqrt{2\pi})^3}\frac{k_i k_j}{k^2}e^{i\vec{k}\cdot\vec{r}} = -\sqrt{\frac{\pi}{2}}\vec{\nabla}_i\vec{\nabla}_j\left(\frac{1}{r}\right). \tag{7.13}$$

Care must be exercised in using the dyadic operator, $\vec{\nabla}_i\vec{\nabla}_j$, on the function $\frac{1}{r}$. The trace of Equation 7.13 is $\int \frac{d^3\vec{k}}{(\sqrt{2\pi})^3}e^{i\vec{k}\cdot\vec{r}} = (\sqrt{2\pi})^3\delta^3(\vec{r})$ where $\delta^3(\vec{r})$ is the usual three dimensional Dirac delta function satisfying $\int d^3\vec{r}\,\delta^3(\vec{r})f(\vec{r}) = f(\vec{0})$ for any well behaved $f(\vec{r})$. Thus we have to separate the trace from the symmetric shinola

In addition, the product of two Fourier transforms is a convolution in configuration space, $\int \frac{dk}{\sqrt{2\pi}}\mathcal{F}(k)\mathcal{G}(k)e^{ik\cdot r} = \int \frac{dr'}{\sqrt{2\pi}}F(r')G(r-r')$, see.