Chapter 8

Covariant Electromagnetic Fields

8.1 Introduction

The electromagnetic field was the original system that obeyed the principles of relativity. In fact, Einstein's original articulation of relativity could justifiably be rephrased as the simple statement: How do we change our concepts of space-time and inertial observers such that Maxwell's equations are the same to all inertial observers? In this sense, there is no question that the Electromagnetism of Maxwell transforms simply under the Poincaré group[1] Our previous treatment of the electromagnetic field dealt with the $\vec{E}$ and $\vec{B}$ fields and the relativistic nature was not obvious. This difficulty is easily removed if we use a set of elements which are manifestly covariant.

To develop this description, it will first be necessary to understand the $\frac{v}{c} \ll 1$, non-relativistic, limit of simple electromagnetic systems. In addition, we can simplify Maxwell's equation by using a more covariant form of units. Our original Maxwell's equations, Equation 5.1, can be made more covariant if we use Heaviside-Lorentz units, described in Section 8.3. We can also develop the language of the fields in four vector form. Since the relativistic manifold has a metric that is not the identity, we will need to formulate the theory using the techniques of general transformation theory, see Appendix G. There is another problem though in that this system has the drawback that the speed of light enters in the units and thus it is not clear how the non-relativistic limit is achieved. This problem will be discussed in

[1]The Poincaré group is the set of transformations that include the Lorentz group and the space-time translations, see Section G.3

Section 8.2.

8.2 The Non-relativistic Limit

Side issue on Gleeson's magnetic paddle

Consider an electron and a large massive magnet, see Figure 8.1. Shoot the electron into the magnet at some speed v. It is deflected and comes out at the same speed that it went in at but moving in the opposite direction. This is very satisfying since the kinetic energy before and after is the same.

Now consider the situation in which the electron is initially at rest and the magnet is moving at the speed v toward the electron. Initially the electron has zero kinetic energy. After it encounters the magnet, the electron is moving away from the magnet at the speed 2v. This is how any massive paddle works. If you hit a light particle with a massive elastic paddle the light object is moving forward with speed 2v.

The striking thing about the magnetic paddle is that like any paddle, the light particle goes from having no kinetic energy originally to one that has kinetic energy. But magnetic fields do not do any work? The proof is simple and direct. The magnetic force on a moving particle of charge e is $\vec{f}_{mag} = e\vec{v} \times \vec{B}$ and the power given the charged particle is $\vec{v} \cdot \vec{f}_{mag} = e\vec{v} \cdot \left(\vec{v} \times \vec{B}\right) = 0$. Yet, the kinetic energy of the charged particle increased from 0 to $2m_e v^2$.

If we analyze the situation in the frame of the moving magnet we see immediately the resolution for this seeming paradox. Since, given the Lorentz force, the only way to change the energy of the charged particle is with an electric field, $\frac{dEn}{dt} = e\vec{v} \cdot \vec{E}$, where E_n is the energy. Whereas in the frame of the magnet, there is only a magnetic field, in this frame, there is not only a magnetic field but also an electric field. In fact, the electric field, $\vec{\mathbf{E}}$, has to be parallel to the plane of motion and thus perpendicular to $\vec{\mathbf{B}}$ and is directed along the sideways displacement that the charged particle experiences. In order to increase the kinetic energy of the charged particle to $2m_e v^2$ the electric field had to be $E = vB$. This result holds for any orientation of the magnet and thus in vector notation $\vec{E} = \vec{v} \times \vec{B}$. The important point is that you can use this paddle to convince yourself that under the Galilean transformation you not only change the coordinates but also have to change the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$. If they are to recover the same laws of physics, what one inertial observers says is a magnetic field will be viewed as being both a magnetic and electric field to another inertial observer.

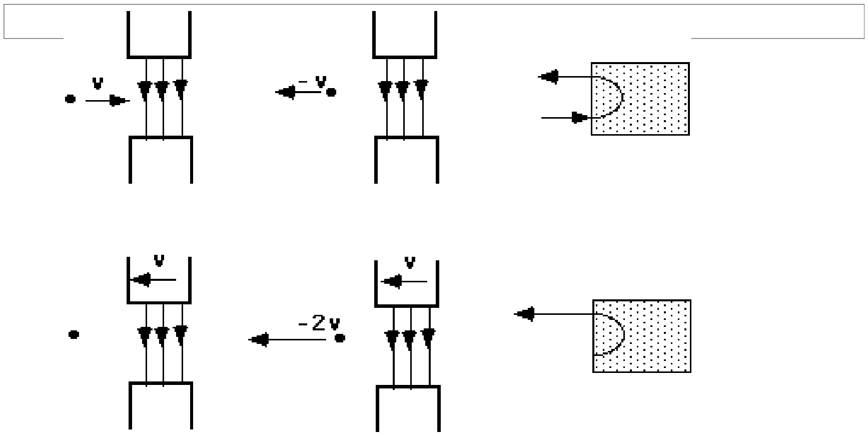

Figure 8.1: **The Magnetic Paddle:** In the upper part of the figure, a small charged particle represented by the dot is moving with speed v into a large magnet. It is deflected and comes out at the same speed v with which it entered. Now consider the same situation viewed from the frame in which the charged particle is initially at rest. Here the magnet is moving with speed v. After the magnet has passed over the original position of the charged particle, the particle is moving to the left at speed 2v.

Another aspect of this discussion is that in this example the electric field is a manifestation of a moving magnetic field. The more usual approach to relativistic electricity and magnetism is to identify that the magnetic field emerges from viewing the electric field in motion. This statement and prejudice is based on the usual identification of the electric current as emerging from charge in motion. From our point of view, there are two problems with this approach. It has been our position that the fields are fundamental objects on their own and do not require the existence of the charges. The other is that the symmetry between the electric and magnetic fields, $\vec{E}' \leftrightarrow c\vec{B}$ and $\vec{B}' \leftrightarrow -\frac{\vec{E}}{c}$, in the source free case is lost. This symmetry is lost in cases with sources because of the absence of magnetic charges. If there were magnetic charges and we suitably modified the Lorentz force, we could reproduce our paddle with a massive capacitor and conclude that $\vec{B} = \vec{v} \times \vec{E}$. It is interesting to note that in Einstein's 1905 paper [Einstein 1905], although it is titled "On the Electrodynamics of Moving Bodies", his analysis of the field transformation properties is for the sourceless and therefore transverse fields. In fact, you realize that the basis for the relativistic transforms is in this segment of Maxwell's equations; this is the basis for our understanding of light in classical physics which, of course, is the basis of our understanding of the need for changes in our understanding of measurements of time and space.

For low relative velocities, $\frac{v}{c} \ll 1$, the fields must transform as

$$\begin{aligned} \vec{E}' &= \vec{E} + \frac{\vec{v}}{c} \times c\vec{B} \\ c\vec{B}' &= c\vec{B} - \frac{\vec{v}}{c} \times \vec{E} \end{aligned} \tag{8.1}$$

To find the high velocity form of the transformation rule, realize that our only requirement in Equation 8.1 is the linearity of the fields. We should also require that the transformation has the combination $\vec{E} \cdot \vec{E} -$ form invariant, see Section **??**Thus the high velocity form must be of the form

8.3 Manifestly Covariant Formulation of E&M

There are several simple substitutions and definitions that easily make the relativistic covariance of the electromagnetic fields, charges, and the potentials manifest. As a first step, we convert to Heavyside-Lorentz units. Here a new unit of charge is defined from the MKS SI units by

$$\begin{aligned} \rho_{HL}(\vec{x},t) &\equiv \frac{\rho_{SI}(\vec{x},t)}{\epsilon_0} \\ \vec{j}_{HL}(\vec{x},t) &\equiv \frac{1}{c}\frac{\vec{j}_{SI}(\vec{x},t)}{\epsilon_0} \\ \vec{E}_{HL}(\vec{x},t) &\equiv \vec{E}_{SI}(\vec{x},t) \\ \vec{B}_{HL}(\vec{x},t) &\equiv c\vec{B}_{SI}(\vec{x},t) . \end{aligned} \tag{8.2}$$

See the discussion in Section 6.3. With these charge units, and using the condition $\mu_0\epsilon_0 = \frac{1}{c^2}$, the Maxwell equations, Equations 5.1, become

$$\begin{aligned} \vec{\nabla} \cdot \vec{E}_{HL}(\vec{x},t) &= \rho_{HL}(\vec{x},t) \\ \vec{\nabla} \times \vec{B}_{HL}(\vec{x},t) &= \frac{\partial \vec{E}_{HL}}{c\,\partial t}(\vec{x},t) + \vec{j}_{HL}(\vec{x},t) \\ \vec{\nabla} \times \vec{E}_{HL}(\vec{x},t) &= -\frac{\partial \vec{B}_{HL}}{c\,\partial t}(\vec{x},t) \\ \vec{\nabla} \cdot \vec{B}_{HL}(\vec{x},t) &= 0. \end{aligned} \tag{8.3}$$

Note that the local charge conservation condition in Heavyside-Lorentz units is

$$\frac{\partial \rho_{HL}}{c\,\partial t}\left(\vec{x},t\right) = -\vec{\nabla}\cdot\vec{j}_{HL}\left(\vec{x},t\right). \tag{8.4}$$

and that the charge density, $\rho_{HL}\left(\vec{x},t\right)$, and the current density, $\vec{j}_{HL}\left(\vec{x},t\right)$ have the same dimensionality. The same situation obtains for the electric, $\vec{E}_{HL}\left(\vec{x},t\right)$, and magnetic fields, $\vec{B}_{HL}\left(\vec{x},t\right)$.

The next step is to identify the Lorentz transformation properties of the fields that enter Equation 8.3. Once the transformation properties are identified, they can be inserted into the usual manifestly covariant forms such as four vectors and second rank tensors, see Section G.3. Obviously, our space coordinates and time variable join to form a contravariant four vector $x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z)$. Also this implies that there is the covariant four vector $\partial_\mu \equiv (\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$.

Here, since we want to deal with the currents, we will have to introduce the potentials and also we will treat the fields in a manifestly covariant fashion. The coordinates are $\left(x^0 = ct, x^1 = x, x^2 = y, x^3 = z\right)$. The degrees of freedom are the four vector potential.

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{8.5}$$

The definitions of the Fourier fields are

$$\mathcal{A}^\mu(k) \equiv \frac{1}{(\sqrt{2\pi})^4}\int d^4x A^\mu(x) e^{ikx} \tag{8.6}$$

with the inverse

$$A^\mu(x) = \frac{1}{(\sqrt{2\pi})^4}\int d^4k \mathcal{A}^\mu(k) e^{-ikx} \tag{8.7}$$

where $d^4k = dk^0 dk^1 dk^2 dk^3$ and $k^0 = \frac{\omega}{c}$ with a similar arrangement for the x's and the range of all the integrals is from $-\infty$ to $+\infty$. The metric is such that

$$kx \equiv k_\nu x^\nu = g_{\mu\nu} k^\mu x^\nu = k^0 x^0 - \vec{k}\cdot\vec{x} \tag{8.8}$$

with a similar notation for other contracted forms, i. e. $k^2 \equiv k^\mu k_\mu$. The reality of the potentials requires

$$\mathcal{A}^\mu(k)^* = \mathcal{A}^\mu(-k). \tag{8.9}$$

The Fourier field amplitudes are

$$\mathcal{F}^{\mu\nu} = -i\left\{k^\mu \mathcal{A}^\nu - k^\nu \mathcal{A}^\mu\right\} \tag{8.10}$$

The equation of motion that results from the action

$$S = -\int \left[\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + j_\mu A^\mu\right] d^4x \tag{8.11}$$

is

$$\partial^\nu \partial_\nu A^\mu(x) - \partial^\mu \left(\partial_\nu A^\nu\right) = j^\mu(x) \tag{8.12}$$

Expressing this for the Fourier potential

$$k^\nu k_\nu \mathcal{A}^\mu(k) - k^\mu k_\nu \mathcal{A}^\nu(k) = -\mathcal{J}^\mu(k) \tag{8.13}$$

where $\mathcal{J}^\mu(k)$ is the Fourier transform of the current density. It should be noted that locality in x space and locality in k space are not the same. In fact, they are almost the opposites of one another: $\delta(x)$, completely local in x, has as a transform $\delta(k) = 1$ that is everywhere in k space.

Often, we work in a Lorenz gauge, $\partial_\nu A^\nu(x) = 0$, then the Fourier potentials satisfy the gauge condition

$$k_\nu \mathcal{A}^\nu(k) = 0, \tag{8.14}$$

and the Fourier potentials satisfy

$$k^2 \mathcal{A}^\mu(k) = -\mathcal{J}^\mu(k). \tag{8.15}$$

We can eliminate the negative frequency fields by using the reality condition, Equation 8.9, and realizing that $d^4k|_{k_0>0} \equiv \theta(k^0)dk^0d^3\vec{k}$ is invariant for orthochronous Lorentz transformations.

$$\begin{aligned}
A^\mu(x) &= \frac{1}{(2\pi)^4}\int \left\{d^4k|_{k^0>0}\mathcal{A}^\mu(k)e^{-ikx} + d^4k|_{k^0<0}\mathcal{A}^\mu(k)e^{-ikx}\right\} \\
&= \frac{1}{(2\pi)^4}\int \left\{d^4k|_{k^0>0}\mathcal{A}^\mu(k)e^{-ikx} + d^4k|_{k^0>0}\mathcal{A}^\mu(-k)e^{ikx}\right\} \\
&= \frac{1}{(2\pi)^4}\int \left\{d^4k|_{k^0>0}\mathcal{A}^\mu(k)e^{-ikx} + d^4k|_{k^0>0}\mathcal{A}^\mu(k)^*e^{ikx}\right\} \\
&= \frac{1}{(2\pi)^4}\int d^4k|_{k^0>0}\left\{\mathcal{A}^\mu(k)e^{-ikx} + \mathcal{A}^\mu(k)^*e^{ikx}\right\} \\
&= \frac{1}{(2\pi)^4}\int d^4k|_{k^0>0}\left\{\mathcal{A}^\mu(k)e^{-ikx} + \text{c. c.}\right\}
\end{aligned} \tag{8.16}$$

where, in the second line, the range of integration from ∞ to $-\infty$ is split and, in the third line, k^μ in the second integral is replaced by $-k^\mu$, and, in the final line, the reality of the field, $A^\mu(x)$ is manifest.

The use of positive frequency Fourier fields is justified by the fact that they are easy to interpret and as we will see they have special dynamical significance. Our Fourier fields satisfy the dynamical equations, Equations 8.13, 8.14, and 8.15. The down side of the use of positive frequency fields is that the limited range of integration upsets the usual completeness arguments of the Fourier expansion.

For many purposes, in particular radiation and when using Green's function techniques, we will be especially interested in source free fields. Setting $\mathcal{J}^\mu(k) = 0$ in Equation 8.13 and not using any restriction on gauge choice, for $k^2 \neq 0$, Equation 8.13 is solved by having the Fourier field, $\mathcal{A}^\mu(k)$ directed along k^μ,

$$\mathcal{A}^\mu(k) = k^\mu \frac{(k\mathcal{A})}{k^2} = k^\mu c(k) \tag{8.17}$$

or

$$k^\nu k_\nu k^\mu c(k) - k^\mu k^\nu k_\nu c(k) = k^2 \left(k^\mu c(k) - k^\mu c(k)\right) = 0 \tag{8.18}$$

and thus with no constraints on $c(k)$.

For $k^2 = 0 \Rightarrow k^0 = |\vec{k}|$ for positive frequency fields. the source free version of Equation 8.13 becomes

$$k^\mu (k^\nu A_\nu) = 0 \tag{8.19}$$

or $k^\nu A_\nu = 0$. In other words, $A^\mu(k)$ is a four vector that is orthogonal in the four vector sense to k^μ. Before proceeding to examine the implications of conditions such as Equations 8.13, 8.14, and 8.15, we must expand the $\mathcal{A}^\mu(k)$ in terms of a convenient four vector basis in k space. For any k^μ, there are four four vectors which can conveniently serve as a basis. One of these is k^μ itself. Since k^μ can be space-like, time-like, and light-like, there is no sense in normalizing it. Firstly, there is k^μ since $k^2 = 0$. The other two are found most easily by considering, for any k^μ, the frame in which $k^\mu = (k, k, 0, 0)$. For that k^μ, the two space-like unit four vectors $e^\mu_{(1)} = (0, 0, 1, 0)$ and $e^\mu_{(2)} = (0, 0, 0, 1)$ are the solutions. For the general case of $k^\mu = (|\vec{k}|, k^1, k^2, k^3)$, we can costruct this general k^μ from the special form $(|\vec{k}|, k, 0, 0)$ by rotating about the z or 3 axis by an angle θ_k so that it becomes $(|\vec{k}|, k\cos(\theta_k), k\sin(\theta_k), 0)$. Then rotate about the x axis by ϕ_k yielding the most general form of

$$k^\mu = (|\vec{k}|, k\cos(\theta_k), k\sin(\theta_k)\cos(\phi_k), k\sin(\theta_k)\sin(\phi_k)). \tag{8.20}$$

Applying this same procedure to $e^\mu_{(1)}$ yields

$$e^\mu_{(1)}(k^\mu) = (0, -\sin(\theta_k), \cos(\theta_k)\cos(\phi_k), \cos(\theta_k)\sin(\phi_k)) \tag{8.21}$$

and

$$e^{\mu}_{(2)}(k^{\mu}) = (0, -\sin(\theta_k), \cos(\theta_k)\cos(\phi_k), \cos(\theta_k)\sin(\phi_k)). \tag{8.22}$$

In addition, the $k^2 = 0$ constraint can be incorporated by defining an new Fourier amplitude, $\alpha^i(k)$ where $i = L, (1), (2)$ such that

$$\mathcal{A}^{\mu}(k) = \left(k^{\mu}\alpha^L(k) + \sum_{i=1,2} e^{\mu}_{(i)}(k)\alpha^{(i)}(k) \right) \delta(k^2) \tag{8.23}$$

Thus, the Fourier potential has the expansion

$$\begin{aligned} A^{\mu}(x) &= \frac{1}{(2\pi)^4} \int d^4k|_{k^0>0} \left[\left\{ k^{\mu}c(k) + \left(k^{\mu}\alpha^L(k) + \sum_{i=1,2} e^{\mu}_{(i)}(k)\alpha^{(i)}(k) \right) \delta(k^2) \right\} e^{-ikx} \right. \\ &\quad \left. + \text{c. c.} \right] \end{aligned} \tag{8.24}$$

with $c(k)$, $\alpha^L(k)$, $\alpha^{(1)}(k)$, and $\alpha^{(2)}(k)$ arbitrary except that for the α terms any $k^2 \neq 0$ are meaningless. Inserting these Fourier potentials into Equation 8.10,

$$\begin{aligned} F^{\mu\nu}(x) &= \frac{-i}{(2\pi)^4} \int d^4k|_{k^0>0} \left[\left\{ \sum_{i=1,2} \left(k^{\mu}e^{\nu}_{(i)}(k) - k^{\nu}e^{\mu}_{(i)}(k) \right) \alpha^{(i)}(k)\delta(k^2) \right\} e^{-ikx} \right. \\ &\quad \left. + \text{c. c.} \right] \end{aligned} \tag{8.25}$$

Note that the two longitudinal terms, $c(k)$ and $\alpha^L(k)$, drop out in the fields and are thus gauge dependent terms of the potential. This is consistent with the observation that source free fields are transverse, see Section **??**. This holds for both the three vector fields and four vector fields.

We can make one further reduction by integrating the light-like constraint, $\delta(k^2) = \frac{\delta(k^0 - |\vec{k}|)}{|\vec{k}|}$ which takes advantage of the fact that the integral runs over only the positive frequency domain. The final form of the fields in terms of the Fourier potentials is

$$F^{\mu\nu}(x) = \frac{-i}{(2\pi)^4} \int \frac{d^3\vec{k}}{2|\vec{k}|} \left[\left\{ \sum_{i=1,2} \left(k^{\mu}e^{\nu}_{(i)}(k) - k^{\nu}e^{\mu}_{(i)}(k) \right) \alpha^{(i)}(k) \right\} e^{-ikx} + \text{c. c.} \right] \tag{8.26}$$

where $k^0 = |\vec{k}|$ throughout.

For completeness, we can give the potential in the case that we are working in a Lorentz gauge, $\partial A = 0$. In this case,

$$X \tag{8.27}$$

Before constructing a solution to the dynamical equations with sources, we should construct the potentials in terms of the of the Fourier potentials in the Lorentz gauge.

The potentials in the presence of sources satisfy Equation 8.12. Here the use of the Lorenz gauge simplifies the situation considerably,

$$\partial^\nu \partial_\nu A^\mu(x) = j^\mu(x). \tag{8.28}$$

In this gauge, the potentials can be solved for with a Lorentz scalar Green's function, Appendix C.3,

$$A^\mu(x^\gamma) = \int d^4x' G(x^\gamma - x'^\gamma) j^\mu \left(x'^\gamma\right). \tag{8.29}$$

where as usual the Green's function depends significantly on the boundary conditions. Analysis in Fourier space is especially appropriate for these Green's function constructions of solutions since Equation 8.29 are convolutions which become simple products in Fourier space.

the propagator satisfies

$$\begin{aligned} \partial^\nu \partial_\nu K^\mu(x^\gamma - x'^\gamma) &= 0. && (8.30) \\ c\partial_{x^0} K^\mu(x^\gamma - x'^\gamma)|_{x^0 \to x'^0} &= \delta^3(\vec{x} - \vec{x'}) && (8.31) \end{aligned}$$

For $k^2 \neq 0$, the solution is

$$\mathcal{A}^\mu(k) = -\frac{\mathcal{J}^\mu(\|)}{k^2}. \tag{8.32}$$

Applying Green's function techniques, we start with