Chapter 2

Measurement

2.1 The Role of Measurement

At the very center of physics is the essential role of experiment. Even the most carefully crafted theoretical system can only be valid if it agrees with experiment. Experiment is the process of careful observation of the world around us. In the process of performing experiments, in some cases, it is possible to control some parts of the activity of observation but that is not the important part of experiment. Experiment is the drawing of coherent information from a situation. In a sense, the idea of theory construction is to develop a method which can consistently bring into a concise set of statements the results of all possible experiments on a given system.

In order to make consistent observations you have to make measurements. Measurement can be both qualitative and quantitative. Often times qualitative measurements can differentiate different ideas of how some process occurs. We will see this in the discussion of the foundations of quantum mechanics, see Chapter 19 on page 403. Most of the time, however, to differentiate competing ideas based on what we see, our observations have to be quantitative. Actually when you think about it, even most qualitative observations are really just very rough quantitative assessments. The reddening of the sky at sunset says a great deal about how the atmosphere works. What do we mean by red in this case. A definite range of values in the wavelength of the light and thus a quantitative assessment. In this sense, all of physics is based on the process of measurement—quantitative observation.

In its simplest form, measurement is basically the comparison of two related things. Whenever a certain circumstance is seen, the 'caused' situation emerges. On both ends of this observation, measurements must be made to know what was set up and what was the result. Therefore, we must understand measurement if we want to understand physics.

A process of measurement is basically a comparison of situations. This process is then formalized by using standards and comparing with these standards. This is best understood when we talk about length. The objects under consideration are separated. After some time, there appears to be a different separation. To quantify this set of events, we can find another separation that does not appear to be changing. An example of separated objects is two scratches on a rigid bar. Comparing the separations under consideration with the separation of the two scratches on the bar allows us to communicate the nature of the new separations. Of course, what is being measured is the length before and after. What most people miss in a discussion of length measurements is the fact that the process of measuring, the identification of a standard and a process is essentially the definition of length. The case of separation measurements is central to our study and available to our experience and that is the example that I will elaborate on throughout this course, but it is also true for all the other cases of measurement. A few other examples of things you can measure are temperature, hardness, intensity of earthquakes, and time. All measurements are of some attribute of a thing that satisfies some common general criteria and the rules for comparing the attribute to be measured and a standard of comparison with the same attribute.

Before going into more detail about these processes, it should be clear that there is a great deal of arbitrariness in this process of establishing the measurement protocol. Not only are there choices of comparison systems, called standards, but there is even an arbitrariness in establishing the processes. It should also be clear that the phenomena under study does not depend on these choices. This arbitrariness will have important ramifications, see Section 2.5 on page 44.

2.2 Measurability

As a first step in developing any system of measurement, we have to agree that the attribute in question is measurable. To be measurable, the attribute must satisfy an objective equivalence or reflexive relationship: if $A \geq B$ and if $B \geq C$ then $A \geq C$. The $\geq$ is an example of areflexive relationship. In other words, a reflexive relationship allows you to establish an ordered set of configurations for the attribute. Once you have an ordered set, you can then map that ordering onto the real line. This is all an abstract way to say that then you can assign numeric values. You will often see us using this trick of mapping an ordered set onto the real line. This action of ordering and assigning a numeric scale is what is meant by setting a standard.

For example, again consider the case of length. If a place A is farther from some selected origin than a place B and another place C is closer than B, then A is farther than C. This is the reflexive part of the act of measuring. This kind of ordering does not work for things like beauty. There are actually two problems with measuring beauty. Firstly, the ordering of objects of art in terms of their beauty is generally not objective and, secondly, several different measures have to be brought together for an assessment. The different measures can lead to different orders. In other words, it is not clear that an objective ordering is possible. In our sense then, beauty cannot be measured.

Once you have established an ordering, you can place values of the measured thing on a numeric scale. That's where the values actually come from. You also have to remember that this mapping onto the real line assumes an underlying continuity that is often there but in some cases may not be. On the other hand, since the points on the real line are dense, if in your ordering, you left something out there is always room to stick something into a gap. All this amounts to saying that the ordering is important and the specific mapping onto the line is not. For many things any other mapping that preserves the ordering is as valid as the one that you are using. There are some ease of use criteria that make some choices better than others. An important one of these is additivity. For instance a distance that is twice as far is assigned a numeric value that is twice a large. In all the attributes quantified to date there has been some sense of combining systems to produce a larger measures. Measures such as distance that can be put on a scale that adds are called extrinsic. Length is extrinsic. Time is extrinsic. Density on the other hand is not. It is said to be is intrinsic. If you take twice as much stuff at the same density you do not have twice the density.

The next step in establishing a measuring system is choosing the standards, see Section 2.3 on page 38. Before I do that though I have to emphasize that regardless of how you establish your standards there is an attribute with a property called measurability. It exists. For our example, length is the measured thing and there are many possible standards and systems but all of them are merely different articulations of the attribute that is length. In this case, we say that there is a dimensional content that is length.

On the other hand, It is also important to realize that all things with the dimensional content of a length are not a "length" in the sense of separation.

In some special circumstances, these quantities can turn out to be a separation but that does not have to be the case. An obvious example is an area. The square root of the area has the dimensional content of a length. This is in the sense that if the area was that of a square, the separation, a length in the fundamental sense, of the corners along an edge is the square root of the area. Another example, you can have the dimensional content of a length when you have a separation which is our prototypical "length", constituted from a speed times a time, or a force times a time squared divided by a mass. In all these cases, there are circumstances in which although they may or may not represent a separation, they are a length. For example in the case of a velocity times a time which has the dimension of a length, this is a length when the velocity is that of an object and the time is a time of flight. For the rules governing the manipulation of dimensional quantities see Section 2.4 on page 43. Another important example of this type is idea of the gravitational acceleration g. g is the gravitational force per unit mass at a place in the vicinity of a massivebody. It is not an acceleration. But the dimensional content is $\frac{\text{force}}{\text{mass}}$ which is the same as a $\frac{\text{length}}{\text{time}^2}$ which the dimensional content of an acceleration in the usual definition as the time rate of change of velocity. In certain circumstances, g is the value that the acceleration takes. For instance, when gravity is the only force acting on the body, g is the acceleration that the body will have. Certainly, if there are circumstances in which g is an acceleration, it must have the dimensional content of an acceleration.

This leads to the next issue. How many dimensional quantities are there? For historical reasons, length, time, and mass are taken to be the primary quantities and things like velocity, a $\frac{\text{length}}{\text{time}}$, are considered derivative. Are there more? As many as you like. To see that let's look at an obvious example. Volume is a length^3 in the sense of the discussion above for area. You find it by multiplying three lengths. At the same time you could have an independent system for the measurement of volumes. For instance the gallon is a measure of volume. You could have a standard gallon and a protocol for measuring volumes based on this standard gallon. In this case, if you can find empirically that a certain number of cubic inches are contained in the standard gallon, 1 gallon = 231 in^3. This would appear as a law of nature and could be called the Law of Volumes. Instead, we use ordinary geometry to conclude that this law is actually a result of our understanding of geometry. This example may seem a little forced but consider a slightly more subtle situation. Consider the case of the inertial and gravitational mass. This will be discussed in great detail later when we look at the problem of General Relativity in Chapter 16 on page 345 but for now we need only know that there are two rather independent properties of mass. We all know that $\vec{F} = m\vec{a}$ and that the mass in this expression indicates how difficult it is to change the velocity of an object. This is called the inertial mass. You measure inertial mass in situations in which objects are accelerated. An alternative concept of mass is the mass that acts to generate the gravitational force. The attractive gravitational force, $\vec{F_{1,2}}$ of one mass, m_1, on a second mass, m_2, is $\vec{F}_{1,2} = G\frac{m_1 m_2}{r_{12}^3}\vec{r}_{12}$, where G is Newton's Gravitational Constant which has the value $6.7 \times 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{s}^2\,\mathrm{kg}}$ in the MKS system and $\vec{r}_{12}$ is the separation vector from body one to body two. This mass would be measured by placing two bodies at a known separation and measuring the force between them. Since these two ideas of mass are so completely different, it is difficult to conceive of why they are given the same name and treated identically. In a very real sense, there are two kinds of mass. We might want to differentiate by calling them by different names which for our discussion will be inertial and attractant. Stuff has so much attractant, a_t or inertial, i_n. You could measure i_n in a situation with a standard force and an acceleration according to $\vec{F} = i_n\vec{a}$. You would also likely define a measurement system for a_t based on the gravitational force but in the form $\vec{F}_{1,2} = \frac{a_{t_1} a_{t_2}}{r_{12}^3}\vec{r}_{12}$ without the use of an empirical constant such as G. In other words, you would say that two bodies of attractant one generated a force of one newton between themselves when placed one meter apart. Then by examining the motion of bodies under the influence of each others gravitational forces discover the empirical law that inertia and attractant are related by $a_t = \sqrt{G}i_n$. Of course, this is not how the subject was developed. Newton realized immediately that objects move under the influence of gravity in a fashion that is independent of their mass and that therefore gravitational and inertial mass are related by $m_{inertial} = m_{gravitational}$ and he never really discriminated between them. The lesson for us is that you can have an independent unit system for anything that can be measured.

On the other hand, it is the practice to consider mass, length, and time as special or primary. In this sense all the other measures are derivative of these three. What would have been empirical relations between measured quantities become definitions such as velocity is the change in separation for a change in time, or in more complex cases become expressed as a law of physics such as $\vec{F} = m\vec{a}$. Why only three and how did we get here? It is a result of the effort of physics to unify all phenomena into as few categories as possible. To classical physicists, thesethree, mass, length, and time, were the irreducible set from which all others could be constructed. We now take a different perspective. There are two ways to look at the modern situation. We have found that as our understanding of nature has improved certain intrinsic quantities have been discovered. For instance, the Special Theory of Relativity has provided a special significance for c, the speed of light. Although it is the speed of light in a vacuum, it is more significant as a measure linking intervals of space and time, see Chapter 9 on page 213. This type of quantity, c, can be used toset a scale of units and these in turn can be used to set scales for length, mass, and time, see Section 2.6 on page 48. In one view you can say that these fundamental dimensional constants provide a basis for a system of measurement as discussed in Section 2.6 on page 48 or they can be viewed as the discovery of new physical law to reduce the number of primary dimensions. In this second view, you could now say that we are down to two and shortly may be reduced to one. Before we are in a position to look at this question closely, we will need to develop some our technical skills for themanipulation of measured quantities in Section 2.5 on page 44.

2.3 Role of Standards

Once you decide that something is measurable, you have to pick a standard for comparison. A standard is something that has the property that you wish to measure. You arbitrarily select the standard and a protocol for using it. For example, for years the meter was the length between two scratches on a bar in Paris. You will obtain different values for the measured quantity depending on the standard of comparison. The distance between Austin and College Station is the same *regardless* of the standard, it is a length, but the *numeric* value depends on the standard or unit system used, miles or feet. There are several criteria for the choice of the standard. It should be convenient, stable, and accessible. Beyond these criteria, the choice can be rather arbitrary.

It very important to again emphasize that **the standard along with the algorithm for comparison, is the definition of the thing that is being measured.** For example, the definition of "hardness" is determined as the quantity you get according to the algorithm stated for finding "hardness". Algorithm "A" is established as the prescription to measure the quantity that will be called "hardness," a specifically shaped diamond needle under a certain pressure moved across the surface of interest. The application of the algorithm to a certain material sets a standard reference that, in partnership with the algorithm, becomes the definition of hardness.

This process can be applied similarly to length and temperature, etc. The "unit" is the name of the particular standard being used. Lengths are in meters or feet; earthquakes are measured in Richters[1].

Since the choice of standard is arbitrary, nothing important can depend on it. The quantities can change but not what happens. This is our first case of a symmetry, a subject that we will discuss at length, see Section 7.1 on page 183. The symmetry under changes of standards, like all symmetries, leads to important consequences. The most important of theseis the useful tool of analysis called "Dimensional Analysis," Section 2.5 on page 44.

It is also important to reemphasize that although we can change the standard, there is still an intrinsic measured quantity; the distance between Austin and College Station is a length; that is its dimensional content of the measured quantity. When we measure the distance we use a specific unit, the mile. We can have lots of units and they are arbitrarily chosen, but we always have a distance whose dimensional content is length. It is useless to state the value of physical quantities without stating the standard that is used to measure them. Conversely, depending on the choice of standard, you can get any value for a quantity and so our sense of big or small. The distance between Austin and College Station is about 100 miles, a nominal distance on our scale. In a distance measure based on atomic diameters, the distance between the cities is huge.

In any measurement, there is also always an accompanying algorithm that establishes a method of comparison. An algorithm is a rule in which all the steps are defined and can be carried out by any person. For length, there is a standard length: the distance between two scratches on a platinum bar stored at the International Bureau of Standards in Paris. The method of comparison for length is to lay a length to be measured next to the standard to see if it is longer or shorter or what multiple or fraction the measured length is. Actually, what we really do is to establish a set of secondary standards that are even subdivisions, or multiples, of the original. This secondary standard is the same in the places where it can be compared directly and then applied in the other domains where the new standard

[1]There is a rather annoying habit among physicists to name things after dead physicists. This practice is most apparent with units of measurement. For example the standard system of units, the MKS or **M**eter, **K**ilogram **S**econd system or **SI** is used throughout the world except the United States, Liberia and Myanmar. Its unit of energy is the Joule named for the physicist who lead in the development of our understanding of processes involving energy. It is more helpful to call the unit of energy in the MKS system the $\frac{\text{Kilogram}\cdot\text{Meter}^2}{\text{Second}^2}$ or $\frac{\text{KM}^2}{\text{S}^2}$ its dimensional content in the MKS system. Better yet, is to call energy a $\frac{\text{Mass}\cdot\text{Length}^2}{\text{Time}^2}$ or $\frac{\text{M}\cdot\text{L}^2}{\text{T}^2}$. See Section 2.5 on page 44

works. This algorithm is useful for medium lengths such as measured on the earth but for astronomical distances and extremely small distances we need an alternatives; you cannot lay a rod down and compare. For situations like this, we find alternative measures which are found to replicate the laying down next to the standard or secondary standard in contexts that we can test and then extend the protocol to those cases for which the laying down process cannot be implemented. For astronomical lengths, we use what is called the standard candle. Basically for certain objects we know how bright some object is close up and use the fact that the brightness decreases with distance in a known pattern and thus from the perceived brightness infer the distance. This approach is used for intergalactic distance measurements.

Actually now, for lengths we don't have a standard. The modern construction for length measurements uses the speed of light and a time for the algorithm. It is simply better to use the definition of the meter as a product of the defined speed for light in a vacuum and a defined time. This has became the standard for all cases, see Section 2.3.1 on page 40.

2.3.1 The Story of Length

The story of length is interesting and pertinent. Length is probably the most basic of measured quantities and its history shows many of the characteristics of all measure systems. The need to measure lengths clearly goes back to antiquity. In particular, measurement of segments of the earth's surface was an important activity even at the time that man was still a hunter gatherer. At best, the distances were measured in crude and qualitative ways. With the advent of agriculture, length measurements took on an even greater significance. Not only was there a need to measure plots of land but there was also a need to standardize the units of measure. In all likelihood, the early measures were a crop yield. The tendency to measure land by yield persisted well into the nineteenth century. This measure was ultimately displaced by the more objective measure based on a predetermined length. As societies became more organized, standards were introduced and managed by the those in control of those societies and the control of the instruments of measure became one of the primary duties of government.

Earlier standards such as the length of the king's foot were a reasonable standard. They could at least be required universally but still they were not stable or convenient when you wanted to use them. At some point, a secondary standard, two marks on a rod, that was made from the primary, the king's foot, became the standard and was kept in a special place. A part of the problem was that there were different kings and different municipalities had different standards. It was so chaotic that in some cases merchants used one length standard to purchase materials and a shorter one by the same name for selling them. It was in this context, that the metric system and the idea of the meter was developed. A solution to the universality and consistency problem.

In 1791, The French Academy of Sciences decided to make a standard of length that was "natural." The hope being that if it was natural it would beuniversal and stable. The need for a better system of measurement was acknowledged by everyone. The Academy was encouraged by the soon to be replaced regime of Louis the XVI and despite the turmoil of the French Revolution was continued by the several new regimes that followed. The Academy choose as the unit of length the meter which which was defined as $\frac{1}{10{,}000{,}000}$ of the quadrant of the Earth's circumference running from the North Pole through Paris. This was an interesting choice because it was difficult to measure accurately and hardly accessible. In some sense it is not even "natural." Because it is not the length of the quadrant but the length of a quadrant of the smooth surface that is at sea level, a quadrant of the geoid, an idealized model of the shape of the earth. At the time of this selection as the meter, a competing idea was to make the standard of length the length of a pendulum whose period was one second. The second at the time being defined as $\frac{1}{60} \times \frac{1}{60} \times \frac{1}{24}$ of the day. This idea was dismissed because of the known variation of g, the acceleration of gravity, and the reluctance to base one fundamental unit on another. The variation of g would require that the meter be defined at one specific location on the earth and the hope was that this standard would be universal and accepted by all nations. The meridian through Paris was chosen not because it was in France but because it provided the longest land mass along a meridian that was in a major country. The problem of the dependence on time for length is interesting in light of our current definition, see later.

It was not long before people realized that the original choice was not a reasonable one. Not only was it hard tomeasure and access, it changes over time. The struggle to measure the meter as defined by the French Academy of Science is an interesting story as told by Alder [Alder 2002]. Also when it was measured later and more carefully, it was wrong. The current best measurement of the quadrant of the geoid is 10,002,290 meters. Although this is better precision than we need for this class, it is not sufficient for a modern industrial society. A new more precise measure is needed. The secondary, the bar in Paris, became the standard.

By 1960 advances in the techniques of measuring the wavelength of the emission lines of atomic radiation had made it possible to establish a more accurate and easily reproducible standard not dependent on any artifact. In 1960, the meter was thus defined in the International System of Units as equal to 1,650,763.73 wavelengths of the orange-red line in the spectrum of the krypton-86 atom in a vacuum. It should also be obvious that these new standards were becoming more precise in order to accommodate the needs of a modern technological society for exacting metrology.

By the 1980s, advances in laser measurement techniques had yielded values for the speed of light of unprecedented accuracy. With the success of the Special Theory of Relativity, see Chapter 11 on page 241, it was realized that the speed at which light in vacuum traveled was a universal constant. It was decided in 1983 by the General Conference on Weights and Measures that the accepted value for this constant, the speed of light, would be exactly 299,792,458 meters per second. The meter is now thus defined as the distance traveled by light in a vacuum in $\frac{1}{299,792,458}$ of a second. This is a subtle but dramatic change in our understanding of length. We no longer use a fundamental distance as the basis for our measure of length. Instead, we use a velocity and a time. Now length is the secondary quantity and length is derivative. This idea can be extended to create a system of units that is based on the Fundamental Constants of Nature, see Section 2.6 on page 48.

2.3.2 Accuracy and Precision of Standards

In the past few years, there have been many changes to the choice of standards. The principle reasons for change has been the need for increased accuracy in measurement. In a modern industrial society, it is essential for successful commerce to be able communicate size in a confident precise manner. In a sense, you can never measure better than your standard can be interpreted. In the section on the "The Story of Length," Section 2.3.1 on page 40, you can ask what is wrong with always using as the definition of the meter the distance between the scratches on a bar at the International Bureau of Standards. This is the definition of the meter and how can another definition be more accurate? When technology advances, and people need to make measurements in the micron and sub-micron range or at astronomical distances, a standard based on scratches on bars cannot be reproducible on these size scales. On the the microscopic scales, where in the scratch is the end of the meter? In a sense, the standard is always accurate and is the definition. But if there is an intrinsic error in the process of reading the standard or if the definition is ambiguous, the definition has only a range of usefulness.

By producing a standard that can be compared with greater precision, all measurements have an improved accuracy. Please note the contrast between the use of the words precision and accuracy in the preceding sentence, see Section 1.3.1 on page 11. In the astronomical case, the comparison algorithm cannot be implemented. There is no way to lay out rods between galaxies. Why not work with an algorithm that can be used? One of the beauties of the use of the speed of light to define length is that the primary standard can be used directly in the measuring process.

2.4 Quantities of Physics

As stated above, most of the quantities of physics are measured. I would go so far as to state that all the important quantities are measured. Since all measurements are comparisons, all quantities have a unit. The lessons of the previous sections are that when you talk about a quantity in physics you always keep track of its dimensional content and when you state a numeric value for a physical quantity, you must also state the unit to which it is compared (i.e., length in meters, mass in kilograms). There are also some non-measured quantities that come from the manipulation of measured quantities. These quantities are dimensionless. There are two sources of dimensionless quantities, mathematical manipulations and cancellation of dimensional content. An example of the first is the "$\frac{1}{2}$" in the formula for the distance moved by an object with constant acceleration, a, in time, t:

$$d = \frac{1}{2} a\, t^2. \tag{2.1}$$

You can also say the same thing about the "2" in the exponent. These quantities are are not measured quantities and there is no sense in discussing their precision and they are dimensionless. They come from the processes of mathematics (the algorithms) that we develop to help us understand important concepts.

Another way that we derive dimensionless quantities is by canceling dimensions. The dimensional content of a compounded physical quantity is algebraic reduction of the dimensional content of the elements of the quantity. In equation 2.1 on page 43, the combination of variables on the right side of the equation, $\frac{1}{2} a\, t^2$, has the dimensional content of the factors composing it, $a\, t^2 \overset{\text{dim}}{=} \frac{L}{T^2} \times T^2 \overset{\text{dim}}{=} L$. Note that I have used the fact that the $\frac{1}{2}$ is dimensionless. In this case, the time dimension dropped out of the term of interest.

Another example in the category of a dimensionless measured quantity is angle.

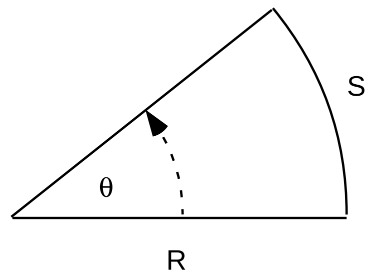

Figure 2.1: **The Radian** The definition of the angle measure called the radian is the ratio of the arc length S to the radius R. The dimensional content of angle is thus $\theta = \frac{S}{R} \stackrel{\text{dim}}{=} \frac{L}{L} \stackrel{\text{dim}}{=} L^0$.

An angle is the ratio of two lengths, see Figure 2.1. It is measured in radians using the ratio of the arc length to the radius for a given opening. In this example, S is a length and R is also a length and, for the angle defined as the ratio, the two lengths cancel out. Angle is a dimensionless quantity.

2.5 Dimensional Analysis

Because you must always maintain the dimensional content of a physical quantity and yet you can measure it in any unit, you obtain a powerful analytic tool called dimensional analysis. The physics behind this is that, since the unit choice is arbitrary, nothing important can dependent on the unit used. This is an example of a symmetry which will be discussed in great detail later, see Section 7.1 on page 183.

Another way to say that you are maintaining the dimensional content is to say that in all relationships involving physics quantities all the terms must be homogeneous in their dimensional content. This is because all the relevant terms of physics are measured quantities and, as stated in Section 2.3 on page 38, all measurements are comparison processes. This is really based on the fact that size is a relative concept; we are large compared to atoms, but atoms are large compared to nuclei. All determinations of measured quantities are a relational operation and large or small is a matter of choice of unit.

We already took advantage of this idea inour discussion of the dimensional content of g in Section 2.2 on page 34. g is the gravitational force per unit mass and has the dimensional content $\frac{\text{force}}{\text{mass}}$. If gravityis the only force acting on a body of mass m, then the force on that body is $f = m\text{g}$ and Newton's Law says that the body with total force f has an acceleration equal to the force divided by the mass, $a = \frac{f}{m}$, or $a = \text{g}$ in that case. Thus although from the definition, g has the dimensional content of $\frac{\text{force}}{\text{mass}}$, if this equation, $a = \text{g}$, is true g must also have the dimensional content of a which is $\overset{\text{dim}}{=} \frac{\text{L}}{\text{T}^2}$. In other words, since the dimensional content can be manipulated algebraically, both of these quantities must have the same dimensional content. For example, a length divided by a time squared has the same dimensional content as an acceleration. An acceleration times a time squared has the same dimensional content as a length. Is it a length? In some cases, it will be, i. e. it is twice the length displaced under constant acceleration, but it is not alength it merely has the dimensional content of a length and only in certain circumstances is it a length.

2.5.1 Uses of Dimensional Analysis

The simplest and most useful application of Dimensional Analysis is the recognition that, since the dimensional content is manipulated algebraically, that you can use it to make sure that your algebraic manipulations are correct. If you havedone a problem asking you to find the time of oscillation of the pendulum of length, l, in the earths gravitational field, g, and you have obtained $T \overset{?}{=} \frac{1}{2\pi}\sqrt{\frac{\text{g}}{l}}$ you can be sure that you made an error because the dimensional content of both sides of the equation are inconsistent. Note that you cannot tell a thing about the correctness of the dimensionless $\frac{1}{2\pi}$ part.

The requirement that the dimensional content of all equations be homogeneous is a lot like the idea that you must only add like things. You can only add apples to apples. You cannot add apples to bananas.

Take, for example, this equation:

$$s = \frac{\text{g}}{2}t^2 + v_0 t \ + s_0 \tag{2.2}$$

Now look at it dimensionally[2]:

$$L \overset{\text{dim}}{=} \frac{L}{T^2} \times T^2 + \ \frac{L}{T} \times T \ + L \tag{2.3}$$

[2]Throughout these notes, in cases where we are only analyzing the results for dimensional content, we will indicate the equation with the symbol $\overset{\text{dim}}{=}$

Using algebraic calculations, we see that each term on the right side of the equation is a length, i. e. $L \stackrel{\text{dim}}{=} L + L + L \stackrel{\text{dim}}{=} L$. This is what is meant by saying that the equation is dimensionally homogeneous, every term has the same dimensional content.

You should get into the habit of checking for the dimensions in an equation. It is a great algebra checker. If you had a formula that said $s = \frac{g}{2}t$, the dimensional content is not homogeneous. Therefore, it is wrong.

Check that the dimensional content of any equation that you write is consistent. It is a good habit to get into.

Probably another place that you have used Dimensional Analysis is in the changing of units. When you are using a given standard as the dimension, then you are using a specific unit. Again, let's examine lengths. Length is the dimension. Several length units are the meter, the foot, and the light year. They are all lengths (L). A neat unit is the "lightnanosecond". It is a length that is about equal to the foot. It is defined as the distance that light travels in one nanosecond. How long is it in inches? In some sense, this is a silly question. It is always the same length. It has different numeric values depending on the unit used. The calculation is simple:

$$1\text{lightnanosecond} = 3 \times 10^8 \frac{\cancel{\text{meters}}}{\cancel{\text{sec}}} \times 10^{-9} \cancel{\text{sec}} \times \frac{39\,\text{inches}}{\cancel{\text{meter}}} = 11.7\,\text{inches} \tag{2.4}$$

You always maintain the dimensional content of all quantities by multiplying by a dimensionless ratio equal in value to one. For example. one foot is 12 inches. Therefore you can multiply any quantity by $\frac{12 \text{ inches}}{1 \text{ foot}}$. An example is the problem of finding how many seconds there are in a year. I will also use this example to show how to do numeric calculations to within a reasonable precision without a calculator. These tricks use the algebraic relations on 'Things'.

Seconds per year:

$$\begin{aligned}
1\ \cancel{\text{yr}} \times \frac{365\ \cancel{\text{days}}}{\cancel{\text{yr}}} \times \frac{24\ \cancel{\text{hrs}}}{\cancel{\text{day}}} \quad &\times \quad \frac{60\ \cancel{\text{min}}}{\cancel{\text{hr}}} \times \frac{60\ \text{sec}}{\cancel{\text{min}}} = 365 \cdot 25 \left(1 - \frac{1}{25}\right) \cdot 3600\ \text{sec} \\
&= \quad 365 \cdot \frac{10^2}{4} \cdot 3600 \cdot \left(1 - \frac{4}{10^2}\right)\ \text{sec} \\
&= \quad 365 \cdot 10^2 \cdot 10^3 \cdot \left(1 - \frac{1}{10}\right) \cdot \left(1 - \frac{4}{10^2}\right)\ \text{sec} \\
&\approx \quad (365 - 36 - 15) \cdot 10^5\ \text{sec} \\
&\approx \quad \pi \times\ 10^7\,\text{sec}.
\end{aligned} \tag{2.5}$$

2.5.2 Scaling Laws

The opposite case of using the dimensional content to check algebra is to use the dimensional content of variables to determine the relationships between the sevariables to produce the relationships. In other words, once you identify the important variables, you must find what combination has the correct dimension. If this combination is unique, then to within dimensionless factors you know the relationships between the variables. These are called scaling laws. Kepler's laws are the direct result of the dimensions of G, the constant from the universal law of gravitation, $f = G\,\frac{m^2}{r^2}$.

$$G \stackrel{\text{dim}}{=} \text{force}\,\frac{\text{distance}^2}{\text{mass}^2} \stackrel{\text{dim}}{=} \frac{L^3}{M \times T^2} \tag{2.6}$$

Suppose you want to know the time (T) that it takes a planet with orbit radius (R) to complete an orbit around a body of mass (M). Since these three are the only variables that can matter, the only combination of these variables with the correct dimension is:

$$T = \sqrt{\frac{R^3}{G\,M}} \tag{2.7}$$

This argument is based on the dimensional content of the variables in the problem. Note that it cannot identify any dimensionless variables such as 2π.

Let's take another example: With one motion of my arm I can throw a ball so high. How much higher will it go if I move my arm through the same motion in half the time? First, break the problem into two parts. (1) To move my arm through the same distance in half the time is to say that I have *doubled* the speed of my throw. (2) How does the height scale with the initial speed? To find the answer, we consider that the only combination of speed and acceleration of gravity (g) that gives a distance is $\frac{v^2}{g}$. So, if I halve the time of the motion (i.e., if I double the velocity), the height will increase bya factor of four. Analysis of this kind is called a scaling relationship.

Another example of simple scaling problem. You are walking with a small child that is $\frac{1}{2}$ your height. Assuming that you are walking in the same fashion with an unforced gait, what is the ratio of your speeds? Answer $-\sqrt{2}$.

The basic idea of this analysis is to identify the relevant variables, and then determine which ones can be combined to form something with the correct dimensions for an answer. In this case we need a speed. This is dimensionally $s \stackrel{\text{dim}}{=} \frac{\text{L}}{\text{T}}$. The relevant variables are the length scale (L) and the acceleration of gravity (g) (this is what is meant by an "unforced gait").

The unique combination of (L) and (g) that is a speed is $\sqrt{Lg}$. Since the length dimension is the height, the ratio of the heights is $\sqrt{2}$. The value of g is unchanged in this example. The situation of the comparing the speed of astronauts on the moon to their gate on earth where the value of g is different from on earth would be a different story.

2.6 Fundamental Dimensional Constants

2.6.1 Sizes

The scale of all things is not arbitrary. In the film "Powers of Ten," Section 1.5.1 on page 24 and in the "Plot of Masses and Lengths," Section 1.5.1 on page 25, we saw that things come in certain sizes. There are no atoms the size of the sun! From our discussion of dimensions, we realize that from the freedom of choice of standards or units that all numeric values of size are possible. What is it then that sets the sizes of things? We also realize that, If the fundamental laws were only expressed by purely mathematical symbols, there would be no factors that could lead to sizes or periods of time. If you want large departures of size using similar rules of the game, you will need to have factors in the rules that reflect the different sizes. These are the dimensional parameters that appear in the equations. These are the determinants of size. Said another way, sizes have to come from somewhere; mathematics cannot provide them.

Let's discuss a concrete example. As discussed earlier in Section 1.5.1 on page 25, all atoms are about the same size. We will discuss this case in detail in Chapter 19 on page 403 but for now all we have to realize is that the size of an atom has to come from the dimensional variables that govern the system. The size of an atom, in particular the hydrogen atom, is set by the fact that it is a system that is composed of an electron held close to a proton by the electric force and using the dynamics associated with quantum mechanics. This says that the size must be determined from combinations of the mass of the electron, $m_e \stackrel{\text{dim}}{=} M$, and the constants associated with the electric forces, $\frac{e^2}{4\pi\epsilon_0} \stackrel{\text{dim}}{=} \frac{ML^3}{T^2}$. You can work this out similar to the analysis of the dimensional content of Newton's Gravitational Constant, G. The use of quantum dynamics brings in Planck's constant. Looking up the units of Planck's Constant in the table of "Things that Everyone Should Know", Section 1.4.2 on page 16, shows that it is an energy times a time, a Joule Sec, and thus has dimensional content of $\frac{ML^2}{T}$. This is a particularly important combination of dimensions and has its own name, Action, which we will discuss in great length later, see Section ?? on page ??. All three of these parameters have dimensional content and there is a unique combination that leads to a length. Work it out. Thus, we see that the size of atoms is set by the parameters that describe the system to within a dimensionless factor which we always assume is of order unity.

Thus we have a rather general result. Although we use mathematics to express our laws, the variables are physical variables and therefore they have dimension. Similarly, in the articulation of any law, there may be and, in general, there will be constant parameters that are themselves dimensional. In a world with no fundamental dimensional constants there would be no scales of size or time. Since we know that phenomena come in specific sizes, fundamental dimensional constants must exist. In any problem, sizes are set by the dimensional parameters of the problem. This means that something in nature is restricting the sizes that we see. Look at the chart of all the things in the universe, Section 1.5.1 on page 25. The things on this chart are concentrated in specific places. This is because of the dimensional constants of the laws governing their behavior: Plank's constant $\hbar$, the gravitational constant G, the speed of light c, the mass of the electron m_e, and the mass of the proton m_p. These constants set the scales of the phenomena that we observe.

Of the family of dimensional constants of nature, some of these are thought to be more fundamental than others. This is in the sense that in some complete theory of everything, all phenomena would be derived from these. The "Fundamental" dimensional constants are $\hbar$, G, and c.

$$\hbar \overset{\text{dim}}{=} M\,\frac{L^2}{T},\; G \overset{\text{dim}}{=} \frac{L^3}{T^2\,M},\; c \overset{\text{dim}}{=} \frac{L}{T} \tag{2.8}$$

This choice is based on the fact that there are indications in our current understanding of nature that all the others will be computed from them or a set that is closely related to them. For instance, in string theory, the latest candidate for a "Theory of Everything," we seem to have a successful approach to a quantum mechanical theory of gravity. The theory has, in addition to $\hbar$ and c, one dimensional parameter, the string tension. The value of the string tension is set once you require that the theory reproduce classical gravity. In this way the tension is set by G. If string theory is to be a "Theory of Everything." then all the masses and strengths of interactions would follow.

2.6.2 Modern Standards

To describe the motion of material objects, there are three independent types of measurements that must be made. All others are combinations of these three. Historically, we used length (L), mass (M), and time (T). Using the laws of physics, all other quantities are then derived from these three fundamental ones. For instance, you may think that there is a measurable thing called force, a push or pull between two bodies. Yes, you could even develop a standard and an algorithm for comparing forces. You might then think that this is an independent unit. But, you also have $\vec{F} = m\vec{a}$. For this equation to be valid for all systems, a force $\stackrel{\text{dim}}{=} \frac{M\,L}{T^2}$, and thus be reduced to the length, mass, and time dimensions. Thus, force can be viewed as just some special combination of our basic units, a derived unit.

Actually though, why couldn't force be fundamental and one of the other units derived? When you think about it you realize that these are obviously a choice. Units are chosen for several reasons: convenience, utility, and reproducibility. If you define length by using scratches on a special bar, it is convenient and reproducible; but, it will not work for extreme cases, so you need to find another method for those cases.

In a discussion of "The Story of Length," Section 2.3.1 on page 40, we have seen that length has now become the derivative concept and that a certain velocity, the speed of light, times a time which clearly has the dimensional content of a length is the defining concept for length. In that sense, now instead of time, mass, and length, we use time, mass, and a special velocity as fundamental standards. In fact, if you think about it you realize that if we use a unit like the "lightnanosecond" mentioned earlier for a length, we are actually working in a system that uses as it fundamental quantities a time and a velocity, the speed of light, instead of a time and a length. If you look at a good table book for the value of the speed of light, you will be told 2.99792458×10^8 m/s (defined). Here we have chosen a speed as our basic unit instead of a length. You define the speed of light to be the appropriate value to reproduce your old standard of the meter as the distance between two scratches on a bar. The bar is now a secondary standard with a finite precision. The primary motivation for the change in definition was the need for increased precision in length measurements. It turns out that it is easier to make very precise measurements of time intervals. Thus defining or better said using as a standard the speed of light and a time you produce a very precise standard of length.

So – What is so special about length, time and mass? The answer is nothing.

Once it is realized that there is nothing special about length, time, and mass there are many options available that may be more useful. What we need is three dimensional entities from which all the others can be found. We know that we need three because the classical system had at least three; length, mass, and time. Our new choice has to be able to reproduce these three. Someday, we might use a time, the speed of light, a velocity, and Planck's Constant, an action. Is this possible? Will it work? In fact, it is likely. Measurements of the Josephson effect involving superconductors would allow a direct definitions of Plancks Constant and thus its use as a defining unit.

Someday, we will may be able to use a velocity, the speed of light c, an action, Planck's Constant h, and Newton's Gravitational Constant, G, the other fundamental dimensisional constants of physics? Probably not. The drive to use fundamental constants comes not from a desire for "naturalness" that so drove the metric choice but from the need for high precision for commercial and scientific applications. The trouble is that it is diddicult to measure G with any precision.

When you think about it you realize that the attempt to base standards on "Fundamental" dimensional parameters is an old one. In the "Story of Length," the French encyclopedists wanted to use the size of the earth. They thought that it would be fundamental. The original definition of mass was based on the density of water–the mass of a given volume of water. The trouble in both these cases is that although these are useful and in principle will work, they are not fundamental and thus always have an intrinsic limit to relevance. We currently reserve the designation "fundamental dimensional constants"only for the three constants $\hbar$, c, and G with the idea that all length mass and time scales will be derived from them. It is our hope that the laws of physics are complete enough that we will ultimately derive all the others from these three. They enter the laws of physics at the most basic level, and we do not expect that we will find a more basic source for them in the future. I am convinced that. if we could produce precise secondary standards from them, we would use a standard system based entirely on them. Our current measurements of $\hbar$ are becoming very precise and, some time soon, we will use it as one of our standards. The problem is that G since gravity is so weak that it may never be measured at the precision required.

Systems of units:

old old	length, density, and time
old	length, mass, and time
new	speed of light, mass, and time
Post modern	speed of light, gravitational constant, and action

Actually the ambiguity in the choice of unit systems is used to simplify calculations. By setting some chosen unit to take a special value, usually one, calculations can take on an especially simple form. The most common place where this is seen is in the Special Theory of Relativity. Many cumbersome c's are eliminated if $c \equiv 1$. This is effectively what is happening when you are using the usual time units, say years, and distance in lightyears. In the end to recover the usual units, you just have to realize whether you are speaking of length or a time. This is carried to an extreme in many computations in which three entities are set to one and all units disappear.

That there are three independent fundamental dimensional constants is not an accident. We expect that they will give us all the structure that we see in the universe. But in the post modern view, almost any three basic independent dimensions will do. In olden times, we had length, time, and mass. If you think about it, you realize that you could as well choose a time, speed, and an energy. The other quantities like length are related to the fundamental constants by the laws of physics. For instance, with a standard force and mass you can derive an acceleration. The modern system uses the speed of light from which time and length are derived dimensions.

Are some choices of unit standards better than others?

We should select those that best fit the criteria–reproducible, available, stable, and precise in the sense that secondary standards are precise. If it turns out that some unit standards are themselves basic laws of physics, then what could be more reproducible? We should use these if they are precise enough for the uses we need.

2.6.3 Useful Dimensions

The following table is a collection of physical quantities, **Quantity**; the vernacular name, **Name**; the usual symbol, **Sym.**; the associated dimensional content, **Dim.** and the SI unit, **SI**. The three dimensional entities are mass, M, length, L, and time, T. The symbols for the SI units for these are m, meter, s, second, and kg, kilogram. Certain SI units are not primary but are honorific. These are the Newton, N, for force which is a $\frac{\mathrm{kg\,m}}{\mathrm{s}^2}$, the Joule, J, which is a $\frac{\mathrm{kg\,m}^2}{\mathrm{s}^2}$ for energy, the Watt, W, for power which is a $\frac{\mathrm{kg\,m}^2}{\mathrm{s}^3}$, the Pascal, Pa, for pressure which is a $\frac{\mathrm{kg}}{\mathrm{m\,s}^2}\ldots$

Table of Useful Dimensions

Quantity	Name	Sym.	Dim.	SI
distance	spatial interval	x	L	m
time	temporal interval	t	T	s
area	area	A	L^2	m^2
volume	volume	V	L^3	m^3
speed	rate of change of distance	v	$\frac{L}{T}$	$\frac{m}{s}$
acceleration	rate of change of speed	a	$\frac{L}{T^2}$	$\frac{m}{s^2}$
angle	opening between two lines	θ	$\frac{L}{L} = L^0$	radian
angular speed	rate of change of angle	ω	$\frac{1}{T}$	$\frac{\text{radian}}{s}$
mass	amount of stuff	m	M	kg
mass density	stuff per unit volume	ρ	$\frac{M}{L^3}$	$\frac{kg}{m^3}$
angular inertia	angular mass	I	$M\,L^2$	$kg\,m^2$
force	mass×acceleration	F	$\frac{M\,L}{T^2}$	N
force moment	torque	τ	$\frac{M\,L^2}{T^2}$	N m
pressure	force per unit area	P	$\frac{M}{L\,T^2}$	Pa
energy	force through a distance	E	$\frac{M\,L^2}{T^2}$	J
energy density	energy per unit volume	u	$\frac{M}{L\,T^2}$	$\frac{J}{m^3}$
power	energy per unit time	P	$\frac{M\,L^2}{T^3}$	W
momentum	mass × velocity	p	$\frac{M\,L}{T}$	$\frac{kg\,m}{s}$
angular momentum	angular inertia × angular speed	**L**	$\frac{m\,L^2}{T}$	$\frac{kg\,m^2}{s}$
action	energy times time	S	$\frac{M\,L^2}{T}$	J s
gravitational constant	strength of gravity	G	$\frac{L^3}{M\,T^2}$	$\frac{m^3}{kg\,s^2}$
Planck's constant	unit of action	h	$\frac{M\,L^2}{T}$	Js
elastic modulus	compressibility	E	$\frac{M}{L\,T^2}$	Pa