Appendix A

Primer on Action

A.1 Newtonian Dynamics

Dynamics is the study of the causes of motion. The motion is the temporal evolution of systems in space. Newtonian physics is based on the idea that space and time are absolute. They are unaffected by what is in it and how it moves. The principles of dynamics were first fully articulated by Issac Newton in his Pricipia

A primary notion is that there are forces. These forces represent the effect of other bodies on the body whose motion is under study. Your third-grade definition of a force–a push or a pull–is as good as any for a start. In this sense, forces are contact actions of one body on another. To do physics, we need to expand this idea beyond contact forces to action at a distance influences, see Section B.1. To get a better understanding of forces, consider the world made up of several parts. This system of parts is isolated and thus all influences are from the parts on each other. This is the essence of reductionism, see Section **??**: you can reduce the whole to its parts and the action of any part on a given part does not depend on the remaining other parts. The important point is that a force is the effect of one body on another and is only considered when you replace the body by its force, see Figure A.1. We are interested in the motion of body one. We talk about the force of body two on body one and the force of body three on body one and so forth. Once we know the forces and use the fact that force in simple cases is a vector quantity and obeys the usual rules for vector addition, we can get the total force by addition. In a real sense, bodies two and three etc. are replaced by their forces. With the advent of a fundamental field theory initiated by Maxwell, see Section 1.3, we have to broaden our idea of force so that it becomes separated from the body that is its source and just talk about it as a thing unto itself. For now, all forces are due to other bodies and they have meaning only in the sense that they are there when we want to discuss the effect that one body has on the other.

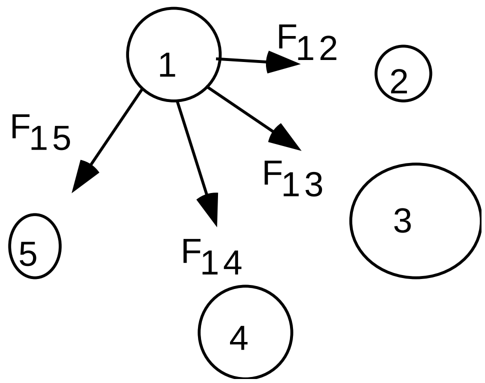

Figure A.1: **Adding Forces** A system composed of 5 parts. The forces are there in the sense that F_{12} is the push or pull on body 1 due to body 2. F_{12} can depend only on the relationship between bodies 1 and 2 and F_{12} does not depend on the presence of the other bodies. Similarly F_{1i} is the effect of body i on body 1. Note also there is a set of forces that act on body 2 and so forth.

The rest of basic dynamics is contained in what are generally called Newton's three laws of motion. The first law states that if a body has no net forces acting on it, it will continue in its present state of motion. This means that the velocity of an unforced body is unchanged; there is no acceleration. Newton took this idea from Galileo, see Section 1.2.

In order to present the second law, we need the concept of mass. For our present purposes, we can take the simple definition of mass: it represents the amount of matter in an object. It was a difficult concept for Newton and the modern interpretations are also subtle. In its simplest form, Newton's second law states the a body responds to the presence of an unbalanced force by accelerating. The acceleration is the net force divided by the mass of the body, the famous $\vec{F} = m\vec{a}$. It is important to note that acceleration is a kinematic quantity and is defined once we have a length and a time.

Newton's third law states that if two bodies exert forces on one another, these forces are equal and opposite. The force of body two in body one is equal to the negative of the force of body one on body two, $\vec{F}_{21} = -\vec{F}_{12}$. This law is also known as the law of action reaction. When this concept of force is a part of the interactions of bodies, this law is always true. With the development of Maxwell's theory of electromagnetism, the idea of action reaction has to be modified, see Section 1.3.

It is very important to realize that, in Newtonian dynamics, if you know the forces acting on a body, either as a function of position or time, and you know the initial position and velocity, then you know the subsequent motion, i. e. the position as a function of time. This is the essence of causality. Given the initial position and velocity, and knowing the forces between all the bodies determines all the subsequent behavior of the bodies. We will find that there is more to the world than just localizable point objects interacting with each other over a distance and that our description of dynamics and requirements for causality have to increase to account for all the phenomena observed in the universe, see Section B.1 and Appendix B.

A.2 Dynamics and Action

Dynamics, as mentioned earlier, are the rules for finding the temporal evolution of a system. In Newtonian Physics, this set of rules was succintly summed up in the rule: $\vec{F} = m\vec{a}$, see Section **??**. In this section, we will find a new way to formulate the rules of dynamics that are more general but still produce the old $\vec{F} = m\vec{a}$ when it is appropriate. The advantage will be that the new rules will work in circumstances in which Newton's Laws were inappropriate or just did not make sense. With these new rules, we will also find a more powerful understanding of the concepts of symmetry and be able to include all the modern constructs needed to clarify the world we live in. We will also be able to use this new procedure to form a more solid understanding of the ideas of energy and momentum. One complication will be that in order to formulate the rule, we will need ideas about kinetic and potential energy that are, in a sense, a throwback to the Newtonian approach. Before we are done, these new ideas will take on a very different and more useful form. We will be able to understand why the massless photon has momentum but first we need to build the necessary background.

A.2.1 Background on Formulation of Action

It is usually not emphasized that the original formulation of Newton's Laws applied to only a very restricted set of circumstances. In Section **??**, Newton's Laws were described as dealing with the effects of one system on another with the assumption that all the parts of the bodies were basically point objects that could move freely in space. This was fine when talking about the planets in orbit around the sun but, even for some of the simplest cases, these conditions do not hold.

Consider the problem of the motion of a blackboard eraser tossed into the air in the front of the lecture hall with a twisting spinning motion. Each part of the eraser is subjected to a huge array of forces. For convenience you can think of the parts of the eraser as the atoms but, even without an atomic hypothesis, all the following considerations still hold. Each part of the eraser is subject to the force of gravity and each part is subject to internal forces from the other parts of the eraser. First, there is an absurd number of parts and forces between the parts and between the parts and the world outside the eraser. We simplify this situation somewhat by assuming that the effect of gravity is the same throughout the eraser and thus reduce these many gravitational forces to a single force acting at one point at the mass weighted center of the body. This is a good approximation for the case of a small eraser in the near vicinity of the earth.

More subtly, we know that, as the eraser twists and spins, the different parts of the eraser will effect other parts. In fact, if the eraser was not a reasonably rigid body and held together by cohesive forces, in the spinning twisting motion, the parts would fly apart. Because the eraser is rigid, there are internal forces that act to hold the respective parts in a fixed relationship to each other. These forces are very complicated. They are in a very real sense unknowable; they are what they have to be to maintain the rigid configuration. These are called constraint forces. The eraser is not an exception. A car on the highway has a constraint force from the road called the normal force that is whatever it has to be to stop the car from falling into the road. Actually, with a little thought it becomes clear that almost all systems have constraints. In other words, the direct application of Newton's laws to systems that are constrained, which is most systems, is wrong or impossible.

In many special cases, fixes were developed that allowed the use of Newton's laws for motion in the presence of constraints and it was well known that this was a problem to both Newton and his immediate followers. The general problem of the motion of systems with algebraically described constraints was solved by Joseph-Louis Lagrange. The procedure that he developed is the modern method for articulating the dynamics of any system and is the one that we will use.

A.2.2 Introduction to Action

The modern approach to dynamics is based on the use of an extremum principle like Fermat's least time theory of light. (Fermat's Least Time theory of light is that light travels between two points in space over the path that takes the least time.) For our case, we postulate that there is a physical quantity called the action and the naturally occurring trajectory in space-time is the one which has the least action. In some sense, this is an unfortunate name for this because we have used the word in another context, see Section B.1, and it has a connotation in the conventional usage. Obviously, before I can make this idea clear, we will need to back up a little as to why and to establish the terminology.

We describe the motion of anything as a connected set of events in space-time, a path in space-time called the trajectory of the particle. The events labeled by a place and a time and are the fundamental entities and a trajectory is a catalogue of the places where the object went as time evolves. an ordered set of contiguous events. Of the infinity of trajectories that can connect two events, the naturally occurring trajectory will turn out to be the one that has the least action.

Consider a piece of chalk tossed up from my hand and returning to my hand some short time later. I am dealing with only one spatial dimension, up. The zero of up is at my hand. The motion of the chalk is a continuous series of events that start with the toss at a time selected to be the zero of time and returns to my hand at a later time T. In between, the chalk has occupied a set of places at specific times between zero and T. If you know the places for all times in that interval you have a trajectory. In Figure A.2, we show the trajectory in a space-time diagram.

Any trajectory is only one of several that have the same total time interval T and start and stop at the same height. Why did nature chose the one that she did? Several possible trajectories are shown in Figure A.3. It will turn out that our rule will be that nature chooses the trajectory from all the possible trajectories that has the least action. Since we have not yet defined the action, this is a little difficult to understand. Not only that but the approach is so different from the Newtonian that we do not have a developed intuition for this way of describing the chosen dynamic .

If you were approaching this problem from the Newtonian point of view, you would have used $\vec{f} = m\vec{a}$ and said that the chalk starts from a given place and given speed. Because there is a force, the attraction of the earth for the chalk, there is an acceleration. Since there is an acceleration, the velocity changes. The velocity changes until it is reversed at the maximum

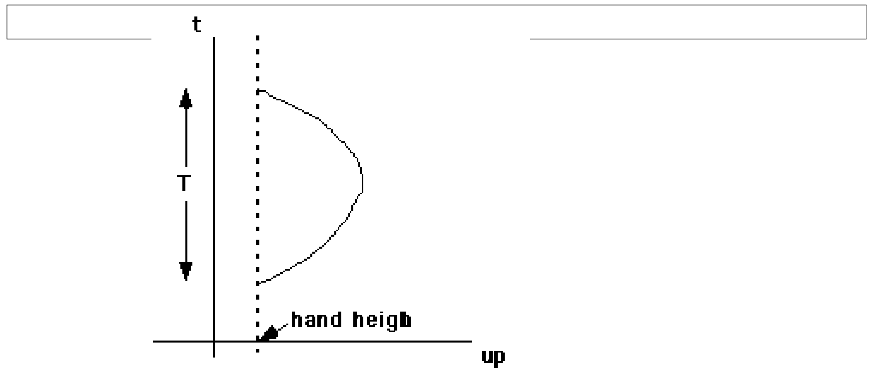

Figure A.2: **Trajectory of a tossed piece of chalk** Chalk tossed from a height labeled zero rises with decreasing velocity until it reaches a peak and then returns to the hand after a time interval T.

height and starts to fall. While all this is happening, the chalk is tracing out a smooth arc in space time. This description is very different than the one that we will be using for action. In the Newtonian formulation, the determination of the trajectory is done at each instant of time at the place at which the chalk is at that time. The action approach on the other hand deals with the action over the entire trajectory. This is a global approach to dynamics. It will be difficult to reconcile these disparate seeming approaches but you have to recover the Newtonian approach for the case in which the chalk can be treated as a point particle and free to move up and down without constraint.

A.2.3 Definition of Action

Instead of $\vec{f} = m\vec{a}$ acting at each point on the body, there is now have a new rule: minimize the action over the trajectory. In other words, nature chooses the least action trajectory from all the trajectories that share the same initial and final event. This is a formulation of motion that is very much like that of Fermat's Least Time formulation for the paths of light in Section **??**. To determine the trajectory, you pick two events, an initial event, x_0 and t_0, and a final event, x_f and t_f. There is a quantity called the action that is computed for every segment of the trajectory. Choose all possible trajectories and the natural trajectory is the one that has the least action.

The action is defined from a function of the positions and velocities

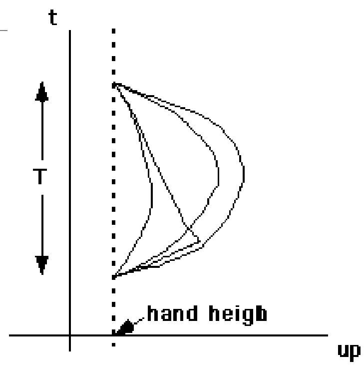

Figure A.3: **Possible trajectories for a tossed piece of chalk** There are an infinity of trajectories that can connect the event at the start of the toss with the event at the return of the chalk to the hand at a later time T.

called the Lagrangian. In this approach to dynamics, instead of trying to figure out what forces are causing the motion, you try to find what the correct Lagrangian is. In a real sense, when a modern physicist develops a new fundamental theory of some phenomena, it is by finding the correct Lagrangian so that the trajectory that yields the least action using that Lagrangian is the one that occurs naturally.

There is a slight technical difference in this case and the case of Fermat's least time. In this case, we create our trajectory segments by creating time slices, see Figure A.4. For Fermat, the segments were sections along the length of the curve. As in the case of least time, the size of the time slices depends on the trajectory and the precision required. This gives a special role to the time variable. Also although we say all possible trajectories, for now, we will only deal with trajectories that advance in time positively. We will be able to lift this condition later, Section **??**.

For a simple point object like the piece of chalk moving up and down, the Lagrangian depends on the position and velocity of the object. Given the Lagrangian, the action is

$$S(x_f, t_f, x_0, t_0; trajectory) = \sum_{trajectory, x_0, t_0}^{x_f, t_f} L(x(t), v(t))\Delta t \qquad \text{(A.1)}$$

Action has the dimensions of an energy times a time. Although this makes the dimensions easy to remember, it is misleading. As we will learn later, the concept of energy is derivative from the action not the other way around,

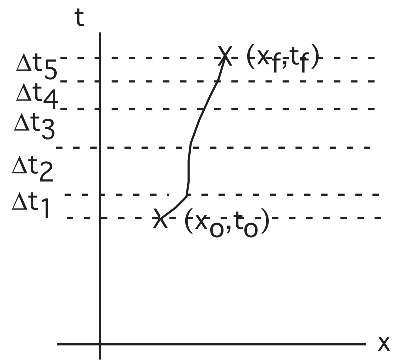

Figure A.4: **Trajectory for the computation of the action** In order to compute the action for a given trajectory, the trajectory is divided into time slice pieces. For each time slice, the positions and the velocity can be determined. The action is then computed for that time slice and the contributions of each time slice are added to produce the overall action. The sizes of the time slices are determined by the rate of change along the trajectory.

see Section A.5 . It would be better to say that energy is dimensionally an action divided by a time. In terms of fundamental dimensional units, the units of action are $\frac{\text{mass}\times\text{length}^2}{\text{time}}$. From Equation A.1, the Lagrangian itself has the dimensions of an energy, $\frac{\text{mass}\times\text{length}^2}{\text{time}^2}$.

The rule that Lagrange found that would reproduce $\vec{f} = m\vec{a}$ for unconstrained systems and also work for more general situations is that the Lagrangian, $L(x(t), v(t))$, should be the difference in the kinetic energy and the potential energy.

$$L(x(t), v(t)) = \frac{mv^2}{2} - V(x) \tag{A.2}$$

where $V(x)$ is the potential energy. Later, Section A.2.5, we will show how this reproduces Newton's laws. It is important to again point out that although this approach requires that you know the kinetic energy and potential energy that these concepts are actually derived from the actions and not the other way. For now, it seems that you need to know the potential energy before you can write the Lagrangian. This is only for historical and pedagogical reasons. When a modern physicist is struggling with understanding some basic new phenomena, it is the other way around. We start with a Lagrangian and then see what the consequences are. It will also turn out that since the actions become the basis of all dynamics, it is the idea that theories that unify other earlier independent theories are considered unified when all the consequences of the theory arise from a single controlling Lagrangean. In modern language, Maxwell unified the electric and magnetic forces because the entire ensemble of equations is derivable from a single Lagrangian and the least action principle.

A.2.4 Trajectory of a Free Particle

To test our new dynamic, let's look at the simplest situation possible – a free particle. A free particle is one that has no forces acting on it. All places have the same energy value and thus $V(x) = 0$. Using Lagrange's rule to get the solution for the free particle in old fashioned physics, we chose the Lagrangian that is just the kinetic energy or $L(v(t)) = \frac{mv^2}{2}$. To make it even simpler, let's require that the released particle is to return to the original position after a time T. The action is

$$S(0,0,0,T,traj.) = \sum_{traj.,0,0}^{0,T} \frac{mv^2}{2}\Delta t \tag{A.3}$$

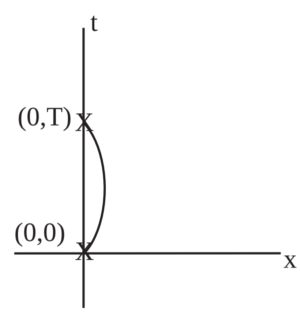

Figure A.5: **Space-time diagrams for the action for a free particle** A particle with no forces acting on it moves between two events,$(0,0)$ and $(0,\mathrm{T})$. A possible trajectory is shown. Our experience with force free motion is that the straight line trajectory is the one that nature chooses; the particle remains at the point of release.

As was stated in the review section, Section **??**, a free particle at rest will remain at rest. Therefore, the natural trajectory for this case is the one that is at the starting place at all times. The is a straight line along the t axis connecting $(0,0)$ and $(0,\mathrm{T})$. How do we obtain this same result using action?

Note that the action is a positive definite quantity for all velocities. Therefore any trajectory that has a non-zero velocity anywhere in the time interval will have a positive action. The trajectory that has $v(t) = 0$ for all t in the interval has an action of zero. This is clearly a minimum of the action since all other trajectories will have a positive action. Thus this is the natural path. Actually any Lagrangian with v^2 in it will accomplish the same thing. The m is in it to give it the correct dimensions and the 2 for historical reasons. In fact, the m that is in the Lagrangian is the definition of mass. More on this later, see SectionSec:Mass.

Using this same result and remembering the material on Galilean invariance in Section **??**, we can solve a more general problem. Suppose we have a free particle that moves through the two events $(0, 0)$ and (x_f, t_f). Again, since the particle is free, the natural trajectory is the straight line connecting these events. To an observer moving by us at a speed of $v = \frac{x_f}{t_f}$, the object is a rest during the entire time interval. To that observer it is free and the initial an final events are $(0, 0)$ and $(0, t_f)$ and the natural path is the straight line along the t axis as before. Thus to us the natural trajectory will be the straight line with slope $fract_f x_f$. Let's obtain this same result with a direct analysis.

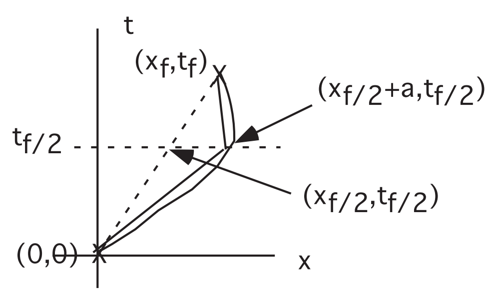

Figure A.6: **Space-time diagrams for the action for a free particle that changes position** A particle with no forces acting on it moves between two events,$(0, 0)$ and (x_f, t_f). A possible trajectory is shown. The general trajectory connecting these events would be very difficult to describe. We will approximate the trajectory with a trajectory that is kinked at the mid-time and straight otherwise.

Consider a general trajectory connecting events $(0, 0)$ and (x_f, t_f), see Figure A.6. Our problem is to find all possible trajectories between these events and then, for each trajectory, find the action. As we discussed about paths when dealing with the Fermat's least time approach to optics in Section **??**. path space is a rich mathematical structure. We want to do analysis. To do analysis we have to reduce the complexity of path space to something that can be described by functions. There are all these same difficulties when dealing with trajectories. To simplify our trajectory space, we reduce the trajectories that we consider to those that are "once kinked". Place the kink along the line $t = \frac{t_f}{2}$, see Figure A.6. In this reduced space, trajectories can be labeled by the distance, a, of the kink from the event $(\frac{x_f}{2}, \frac{t_f}{2}$ along that line. Using this trajectory in the appropriately modified Equation A.3 to take account of the new ending event, and the fact that the inverse slope of the line is the velocity in that segment, it is easy to compute the action for the trajectory labeled a. It is

$$S(0,0,x_f,t_f,traj = a) = \frac{m}{2}\left(\frac{\left(\frac{x_f}{2}+a\right)^2}{\frac{t_f}{2}} + \frac{\left(\frac{x_f}{2}-a\right)^2}{\frac{t_f}{2}}\right). \qquad \text{(A.4)}$$

This is an even function of a and thus has a minimum at $a = 0$. This confirms our result that the natural trajectory, the constant velocity trajectory, is the least action trajectory.

A.2.5 Proof that the Least Action Reproduces Newtonian Physics

See Feynman's famous lecture. It was handed out in class

A.2.6 Examples of action – gravitation near a flat earth

As a simple example that we are all familiar with, consider the case of motion above the surface of the earth. Here the energy of position, the potential energy, is due to the gravitational interaction of a massive body with the earth. For this case, the potential energy at a height h above the earth is $V(\vec{r}) = -\frac{Gm_em}{R_e+h}$, where m_e is the mass of the earth, m the mass of the body, and R_e is the radius of the earth. For motion near the surface, a few meters up or down, from "Things Everyone Should Know," Section **??**, we can use $(1+x)^n \approx 1 + n\ x$ for $x \ll 1$ to reduce this to

$$V(h) = -m\left(\frac{Gm_e}{R_e}\left(1 - \frac{h}{R_e}\right)\right) = V(R_e) + mgh,$$

where we recognize $g = \frac{Gm_e}{R_e^2}$. Since this potential is to be used in an action, as we will see later in Section A.5, changing the action by a constant does not change the physical results in a significant way, we can drop the $V(R_e)$ term.

This reduces the potential energy for objects moving in the near vicinity of the earth to

$$V(h) = mgh. \tag{A.5}$$

Another way to look at this result is to say that for motion restricted to be near the surface of the earth, the earth appears as an infinite plane. In this case, the force of gravity above the plane can not depend on anything, in particular, the height above the plane or the position sideways over the plane. Thus the force also can only be toward or away from the plane. Then realizing from the analysis above in Section A.2.5 that the change in potential as you change position is the force, the only form for the potential in this case is $mgh +$ constant.

For now let us consider only up and down motion, not any sideways motion. The potential energy is mgh where h is the height. Thus the action for any trajectory between an initial height, h_0 at time t_0 and final height, h_f at time t_f is

$$S(h_0, t_0, h_f, t_f; traj.) = \sum_{traj., h_0, t_0}^{h_f, t_f} \left(\frac{mv^2}{2} - mgh \right) \Delta t \tag{A.6}$$

where the path is given by $h(t)$. Note that if you know $h(t)$, you also know $v(t)$. You can see from the form of the action that you will lower the action by having $h(t)$ to be at large h for as much time as possible. The problem is that since the initial and final position and time are given, it takes high velocity to get to large h. The high velocity increases the action. $\Longrightarrow$ There is a single least action path. This is the trajectory that the particle follows.

Let's get more specific. This is again the problem of a piece of chalk tossed up in the air. First the simplest case, the chalk is released and returns to the same height after a t time T.

We need to study the action for all trajectories connecting these events. Again, because of the complexity of the idea of all trajectories, we will need to reduce the number of trajectories. A first step is to use our experience to limit ourselves to simple trajectories that rise smoothly to a peak at some height a at which time the velocity is zero and then returns over a trajectory that is a reflection of the one on the rise. Our natural trajectory must be in that family. This is still a very rich family and too rich to do analysis. This is the same problem that we had with the Fermat's Least Time, Section **??**, and the free particle, Section A.2.4. As in the latter case, the once kinked path can be used to approximate the family of smooth trajectories that have these properties, see Figure A.8. Here again the variable a is the height of

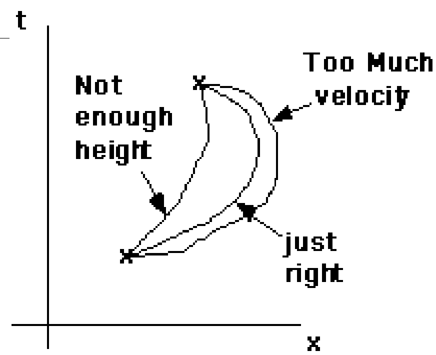

Figure A.7: **Trajectory for Particle in Uniform Gravitational Field** Space-time diagrams for calculation of the action for a particle in a uniform gravitational field. The least action trajectory is just the right compromise between too much kinetic energy and some potential energy.

the approximate trajectory but more importantly now it is a label that can be used to specify the particular trajectory from the family with which we are dealing.

Since this approximate trajectory is broken line segments, it is relatively easy to compute the action.

$$S(0,0,0,T;traj.) = \sum_{(0,0)\ \text{traj.}}^{(0,T)} \left(\frac{mv^2}{2} - mgh\right)\Delta t. \tag{A.7}$$

For a straight line path, v is a constant and is the inverse slope of the line, and is $\frac{a}{\frac{T}{2}}$ in magnitude for both segments. The height is a more subtle question since it varies with time from 0 to a. Being reasonable, we can use the average height, $\frac{a}{2}$. For the sophisticates among you, there is the problem that the concept of average is a not trivial, see Section **??**. Thus the action for the first segment is

$$S_1(T,a) = \frac{ma^2}{2\left(\frac{T}{2}\right)^2}\frac{T}{2} - \frac{mga}{2}\frac{T}{2}. \tag{A.8}$$

Note that once I have made a mapping of the paths onto the line that S becomes a regular function of the path label, a, instead of a functional. Although the velocity is negative, since only v^2 enters the lagrangian, the action on the second segment is the same and the total action is

$$S(T,a) = 2S_1(T,a) = ma\left(2\ \frac{a}{T} - \frac{g}{2}T\right) \tag{A.9}$$

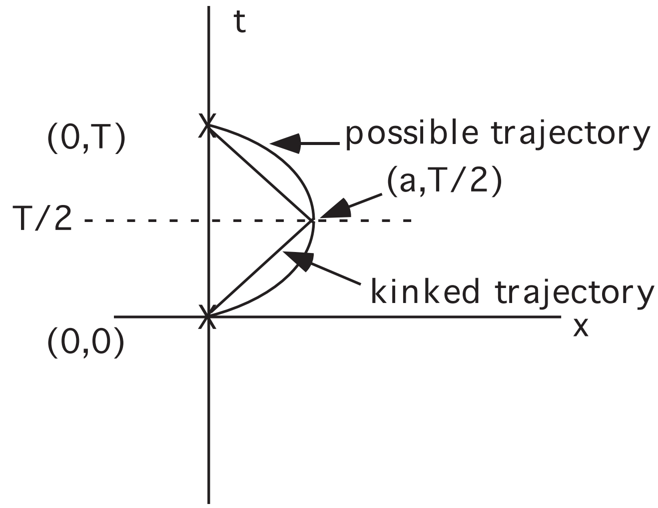

Figure A.8: **Possible trajectory for the action for a particle in a uniform gravitational field** A piece of chalk is tossed upward and caught later at the the same height. A possible trajectory is shown. The natural trajectory is one from the family of smooth trajectories that rise to a peak at a height a smoothly and then return to a lower height on a reflected trajectory. This is still a large family of trajectories. We can approximate the members of this family with a once kinked trajectory with the same height at the time $\frac{T}{2}$.

This has zero's at $a = 0$ and $a = \frac{gT^2}{4}$. The dependence of the action on the path label a is shown in Figure A.9. I have used dimensions in which $g = T = 1$.

We can see that there is a minimum half way between the two zero's at $a = 0$ and $a = \frac{gT^2}{4}$. This implies that the trajectory from this set that is the least action trajectory is the one with

$$a_{least\ action} = \frac{gT^2}{8}. \tag{A.10}$$

Since this is not only the path selecting parameter but is also the height, we get that the height is $\frac{gT^2}{8}$.

A.2.7 Same Example done another way

I am going to do some mathematics here that I do not expect that you will be able to reproduce. I do this to show you that it can be done and that the

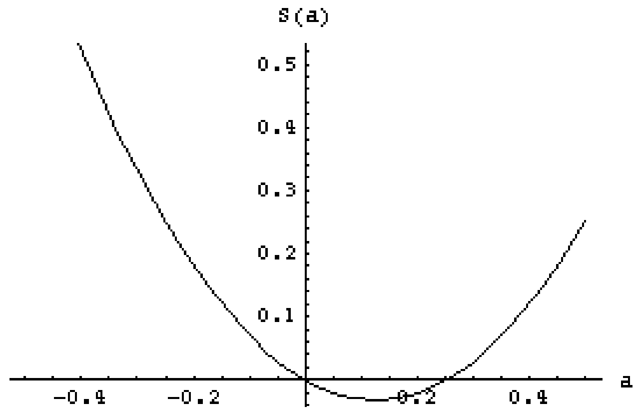

Figure A.9: **Action as a function of** a The action as a function of of the trajectory label a. This curve is a combination of a parabola, $\frac{2m}{T}a^2$, concave up with its vertex at the origin and a straight line, $-\frac{mgT}{2}a$, with negative slope through the origin.

ideas of mathematics are useful. You are not expected to do integrals and take derivatives although you should be able to follow a development using them.

Once again, we want to examine the case of an object of mass m moving in the vicinity of the earth. We can also guess that the correct answer for the height as a function of time is a parabola, all parabolas that fit the time interval are of the form $h(t) = at(t-T) \Rightarrow v(t) = 2at - aT$, where a is label of the path in path space. In this case, a has the dimension of an acceleration, $L \stackrel{\text{dim}}{=} a \times T^2$ or $a \stackrel{\text{dim}}{=} \frac{L}{T^2}$.

The Lagrangian is $L = \frac{1}{2}mv^2 - mgh$ and the action is

$$\begin{aligned} S &= \int_{(x_0,t_0),\text{Path}}^{(x_f,t_f)} \Big(\frac{1}{2}mv^2 - mgh\Big)dt \\ &= m\int_0^T \Big(\frac{1}{2}(2at-aT)^2 \; - gat(t-T)\Big)\,dt \\ &= m\Big(\frac{a^2T^3}{6} + \frac{1}{6}agT^3\Big) \end{aligned}$$

This can be factored to $S = \frac{mT^3}{6}a(a+g)$.

To find the minimum, we can again realize that there are two zeros of S(a). One at $a = 0$ and one at $a = -g$. The minimum is half way between them at $a_{least\ action} = -\frac{g}{2}$

Otherwise, we can take the derivative of S(a) with respect to a and set it equal to zero. Thus

$$\begin{aligned}\frac{dS}{da} &= \frac{d}{da}\left(\frac{mT^3}{6}a(a+g)\right) \\ &= \frac{1}{6}amT^3 + \frac{1}{6}(a+g)mT^3 \\ &= \frac{1}{6}\ (2\ a+g)\ m\ T^3 \qquad \text{(A.11)}\end{aligned}$$

or $a_{least\ action} = -\frac{g}{2}$ is the natural trajectory. In Figure A.10, note how the action varies with a. Again I have used units with $g = T = 1$.

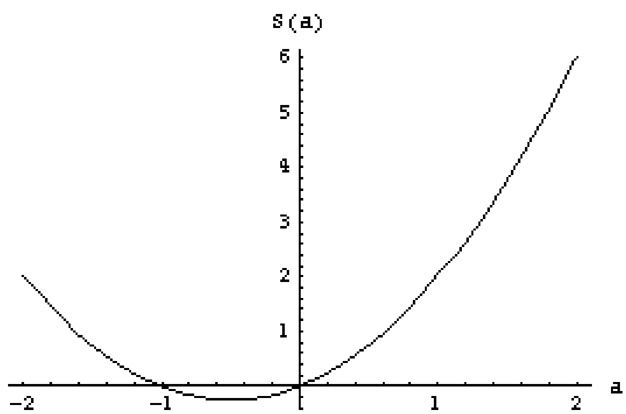

Figure A.10: **Action as a function of** a **as an acceleration** Action as a function of a when the parameter a has the dimensions of an acceleration. This example shows that the trajectory label does not have to be a height.

A.2.8 More Examples of Actions

Scattering

Two particles, one of mass m_1 and the other of mass m_2 collide. After the collision, the particles move away from each other, both still with masses m_1 and m_2. This is a very special problem whose important cannot be over emphasized. In a very real sense, when we probe the nature of the elementary constituents of matter, scattering experiments are the primary source of our knowledge. In addition, the process is so basic that it will allow us to begin to better understand many fundamental issues.

How do we handle this process? First, we have to decide what is meant by two independent particles. Before the particles make contact, they move as if the other particle was not present, i. e. they are independent. It is reasonable therefore to assume that while they are apart or not interacting, the two particles actions add and are the usual free particle action. In other words, there is a free particle action the tells you all the properties of what is meant by a particle and its nature. For our construction of the action of the free particle in Section A.2.4, we used the Lagrangian $L(x, v) = \frac{mv^2}{2}$. The Lagrangian says the the object identified as a free particle does not treat different places differently and thus there is no x dependence in the Lagrangian. If we want to recover Newton's Law, see Section A.2.5, we use the usual classical kinetic energy. We will find that in other circumstances, for instance for a rapidly moving particle, Section 7.2, that a different free particle Lagrangian is appropriate. If we wanted to describe something more complicated than a point particle, say a small rod, we would need elements that deal with what a rod is such as moment of inertia and directional variables.

By using as the action the sum of the single particle actions, the properties of the total system will be the sum of the properties of the parts. If we did this though, and this was the end of it, nothing interesting would ever happen; the particles would merely pass through each other unchanged in their motion. We want them to scatter. Thus in addition, we need to add a part that carries the interaction. The interaction will have a Lagrangian that is made up of relationship variables such as their separation in addition to the particle labels. In other words, the action is made up of the following parts:

$$\begin{aligned}\text{Total Action} &= \text{Free Action(variables particle 1)}\\ &+ \text{Free Action(variables particle 2)}\\ &+ \text{Interaction Action(variables particle 1,}\\ &\quad\ \text{variables particle 2, relationship variables).} \qquad \text{(A.12)}\end{aligned}$$

Of course, it is actually redundant to list the relationship variables in the interaction action since they will be composed of the variables of particle 1 and 2 anyway. The importance of displaying the relationship variables separately is to be able to say that, for a scattering situation, the interaction action is zero when the relationship variables such as the separation are large. In a collision, we assume that most of the time the particles travel toward or away from each other and that the interaction terms contribute only for a short time when the particles are in contact and thus this interaction term is small and does not add significantly to the total action of the process. Another point to note is that, since the interaction terms are dominated by the relationship variables, the contribution from the interaction action should be independent of where and when the collision takes place. Thus, we can write the action for this simple one dimensional scattering process as

$$S = \sum_{(x_{10},t_{10}),Path}^{(x_{1f},t_{1f})} m_1 \frac{{v_1}^2}{2} \Delta t + \sum_{(x_{20},t_{20}),Path}^{(x_{2f},t_{2f})} m_2 \frac{{v_2}^2}{2} \Delta t + A, \tag{A.13}$$

where A represents the interaction action. The scattering process is shown in Figure A.11.

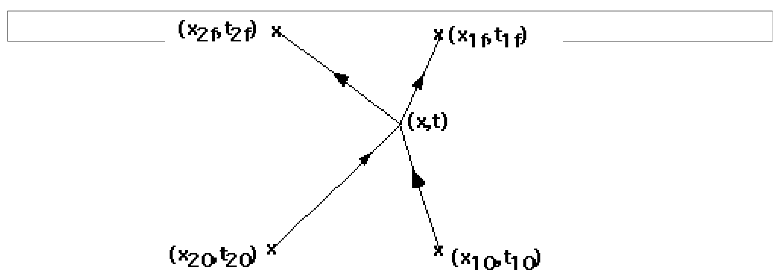

Figure A.11: **Space-time diagram for a scattering event** Two particles of mass m_1 and m_2 free to move in one spatial dimension are directed at each other and collide at the event (x, t) and then move apart . A space-time diagram for a scattering event with particle one starting at event (x_{1_0}, t_{1_0}) and returning to (x_{1_f}, t_{1_f}) and particle two starting at event (x_{2_0}, t_{2_0}) and returning to (x_{2_f}, t_{2_f}) is shown. Although all trajectories connecting the initial and final events and the collision event should be examined, we know that free particles have a natural trajectory that is a straight line, see Section A.2.4.

We want to do all paths but we know that the straight path is the least action for a free particle and so all we need to do is use straight paths between the initial and collision and collision and final events. We can immediately write down the action as a function of the position and time of the collision. The coordinates of that event are the only free parameters in the problem.

Note that we are being consistent in our use of action. When you talk about collisions in the general physics class you set the initial velocities. Here we use the initial and final events. Evaluating the free particle actions, for this system of trajectories, the action is

$$S = \frac{m_1}{2}\frac{(x - x_{1_0})^2}{(t - t_{1_0})} + \frac{m_2}{2}\frac{(x - x_{2_0})^2}{(t - t_{2_0})} + \frac{m_1}{2}\frac{(x_{1_f} - x)^2}{(t_{1_f} - t)} + \frac{m_2}{2}\frac{(x_{2_f} - x)^2}{(t_{2_f} - t)} + A. \tag{A.14}$$

We want to find the trajectory that has the least action and since we have now reduced the world of trajectories to the label of the collision point, x and t. Thus we need to minimize this in what are now the labels, x and t. You could plot this and find the minimum by hand , see Figure A.12, but, if you allow me to use calculus, I can find a simple analytic expression for the $x = x_{min}$ and $t = t_{min}$ that yields the least action. This means taking the derivatives with respect to x and t and finding the value of x and t that satisfy $\frac{\partial S}{\partial x} = 0$ and $\frac{\partial S}{\partial t} = 0$. This x and t label the naturally occurring trajectory.

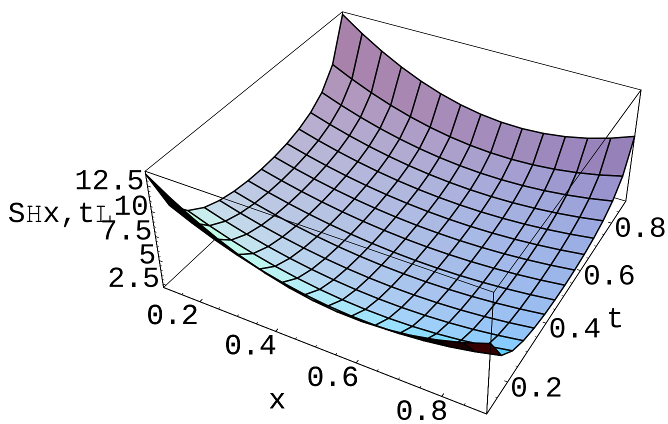

Figure A.12: **Action for a Scattering Event** Action as a function of x and t for a scattering event shown in Figure A.11. There is a clear minimum and it occurs at the points at which Equation A.16 and Equation A.18 are satisfied.

Take my word for it. The condition for a minimum in x is

$$m_1\frac{(x_{min} - x_{1_0})}{(t_{min} - t_{1_0})} + m_2\frac{(x_{min} - x_{2_0})}{(t_{min} - t_{2_0})} - m_1\frac{(x_{1_f} - x_{min})}{(t_{1_f} - t_{min})} - m_2\frac{(x_{2_f} - x_{min})}{(t_{2_f} - t_{min})} = 0 \tag{A.15}$$

or

$$m_1 \frac{(x_{min} - x_{1_0})}{(t_{min} - t_{1_0})} + m_2 \frac{(x_{min} - x_{2_0})}{(t_{min} - t_{2_0})} = m_1 \frac{(x_{1_f} - x_{min})}{(t_{1_f} - t_{min})} + m_2 \frac{(x_{2_f} - x_{min})}{(t_{2_f} - t_{min})} \tag{A.16}$$

Realizing that momentum is mv in classical physics and that v is the difference in positions divided by the the differences in times, this is the statement that the momentum into the collision is equal to the momentum out of the collision.

The condition that there is a minimum in t gives

$$\frac{m_1}{2} \frac{(x_{min} - x_{1_0})^2}{(t_{min} - t_{1_0})^2} + \frac{m_2}{2} \frac{(x_{min} - x_{2_0})^2}{(t_{min} - t_{2_0})^2} - \frac{m_1}{2} \frac{(x_{1_f} - x_{min})^2}{(t_{1_f} - t_{min})^2} - \frac{m_2}{2} \frac{(x_{2_f} - x_{min})^2}{(t_{2_f} - t_{min})^2} = 0 \tag{A.17}$$

or

$$\frac{m_1}{2} \frac{(x_{min} - x_{1_0})^2}{(t_{min} - t_{1_0})^2} + \frac{m_2}{2} \frac{(x_{min} - x_{2_0})^2}{(t_{min} - t_{2_0})^2} = \frac{m_1}{2} \frac{(x_{1_f} - x_{min})^2}{(t_{1_f} - t_{min})^2} + \frac{m_2}{2} \frac{(x_{2_f} - x_{min})^2}{(t_{2_f} - t_{min})^2} \tag{A.18}$$

Which is the same as the statement that the energy into the collision event is equal to the energy out of it.

Figure A.12 shows the action as a function of the position and time of the collision event. This is for the case that $\frac{m_2}{m_1}$is 1.5 and the original and final events for particle 1 are (0,0) and (0,1) and for particle 2 are (1,0) and(1,1).

This exercise also gives us an interesting insight on what mass is. In an early assignment in this course, you were asked to devise a method for measuring mass that does not relay on gravity. Some of you came up with the idea of using collisions to define a mass scale. You can see that this analysis is directly relevant to that kind of definition. In the construction of the action, for the case of the single particle, mass is an overall factor; it is the thing you put in front of the v^2, in the action. If the world consisted of only one particle, mass would be irrelevant since all it does is multiply the action. The process of finding the natural trajectory is unchanged by the an overall scale factor on the action. Mass becomes interesting only when you have more than one particle. If there is more than one particle, you can not remove all the masses with a single scaling factor. The ratios of the mass remain. Consider a scattering event between two particles with the initial and final positions of the two particles the same before and after the collision. If the particles had equal masses, the position of the collision event is at the center. The trajectories of both particles are equally kinked. On the other hand, the higher the mass ratio of say the second particle, the less the trajectory associated with that particle will kink when it collides with another particle. In the limit of a very large mass second particle, there is no bending of the second trajectory and it looks like the first particle has hit a brick wall. This is the essence of inertia.

A.3 The Nature of Symmetry in Physics

In many respects, symmetry in physics is very similar to that in art; there are families of transformations that lead to unimportant changes in the situation. The differences deal with the things on which the transformations act and the definition of unimportant. As expected, in addition, the language that described the actions are more precise and abstract. We will also categorize the transformations of physics in a formal way and use these labels to describe important results.

A.3.1 Discrete Transformations

These are changes that can only be applied in discrete steps. Bilateral or mirror symmetry about a plane is an example from art. For the snow flakes, the rotations at $\theta = n\frac{\pi}{3}$ for $n = 1, 2....$ is an example of a family of discrete transformations that produce a symmetry. What do you think happens for $n = 0$? Is this the same as $n = 6$? The rule is that, once you have a set of transformations, the set must contain all combinations of the transformations for the set to be complete.

The example in physics that corresponds to bilateral symmetry is called a spatial inversion which is to replace places in one directions by their opposite. In a world with on space dimension, replace x by $-x$. In a world with three spatial directions, replace (x, y, z) with $(-x, y, z)$. This is like placing a mirror in the plane $y = 0, z = 0$. This is obviously a discrete transformation. You also note that, if it is applied twice, there is no change. It is said to be a discrete transformation of cycle two; it has two elements, do nothing, the identity transformation, and the inversion. There are many discrete transformations of cycle two: if you have identical particles, you can interchange the particles, you can invert the time, you can do a spatial inversion along the y or z axis, ...

There are, of course, discrete transformations with cycles higher than two. The snowflake example from art carries over to physics. Rotations about the origin by an angle of $\frac{2\pi}{n}$ is an example of a discrete transformation with n cycles.

You can also have a family of discrete transformations that have an infinite number of elements. In one spatial dimension, you can shift the origin by a fixed amount, a. You can do this any number of times generating a set of transformations that has a countable infinite number of members.

It is important to realize that the method by which the members of a family of discrete transformations are labeled must itself be a discrete set of labels and that the members of a discrete set of transformations cannot be labeled by a continuous variable.

A.3.2 Continuous Transformations

Continuous transformations are changes that can be applied for arbitrarily small changes. The labeling of the transformations is a continuous parameter. Rotations about a point are a valuable example. In art, a world of concentric rings would enjoy a symmetry for rotations about the center point. These changes in angle can take any value from zero to 2π. This idea is carried over to physics. In a three dimensional space, rotations about an axis are a family of transformations. These transformations are an example of continuous transformations. Other obvious examples are translations in space and time. Changes in the scale of length discussed in Sections **??**, and **??** is also a continuous set of transformations. **Again it is important to realize that a continuous family of transformations can only be labeled by a continuous variable.**

It is possible to make a discrete family of transformations from subsets of continuous transformations such as the set of rotations used in the snowflake example of Figure **??** in Section **??**. Of course, the reverse process is not possible; you cannot make a continuous family of transformations from a subset of a discrete family no matter how large the set of discrete transformations.

A.3.3 Identity Transformation

The identity transformation is the one that leaves everything alone. The example $n = 0$ in the discrete case above is an identity transformation. Note that $n = 6m$ where $m = 1, 2, 3...$ is also the identity and we already had it in the set of transformations. In fact, any transformation in which $n > 6$ is the same as the transformation $n' = \mathrm{mod}_6(n)$.

A.3.4 Examples of symmetry in situations like physics

You are planning a trip between Austin and College Station. There are several routes.

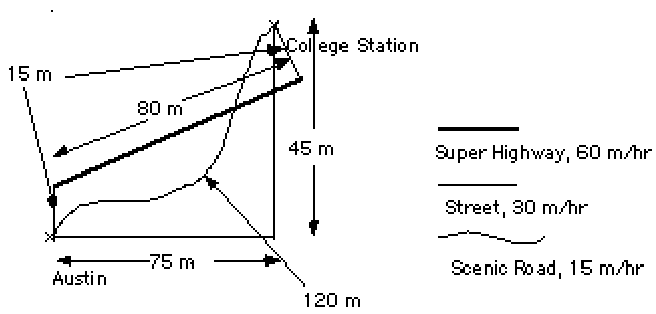

Figure A.13: **Paths to Texas A&M** Miles to AM

A.3.5 Physics transformations:

There are several criteria that you can use to select the route: least time, least distance, see most trees and hills - one hill is worth a dozen trees. There are several changes that you can make in the system: interchange Austin and College Station, interchange super highways and streets, make the speed limit 50 $\frac{m}{hr}$, measure all distances in feet. These are all discrete transformations. You could shift the entire thing a distance x to the east and we all know that as you go east there are no longer any hills. You could shift all the distances by a scaling factor α. These are continuous transformations. For all of these you can see if the transformation effects the evaluation of the criteria.

From this example you see that you need both a set of transformations and a criteria.

A.4 Examples of Symmetry in physics

In physics we are interested in what happens to things in space time, i. e. events. These are labeled by (x,t). An event is a point in a space time diagram. A connected set of events is a trajectory. This is the path that a particle follows as it moves. This is often called a particles world line.

A.4.1 Physics transformations:

Space Reflection:

This is the transformation that corresponds to the bilateral transformation that we discussed earlier. We reflect all the events through the line $x = 0$

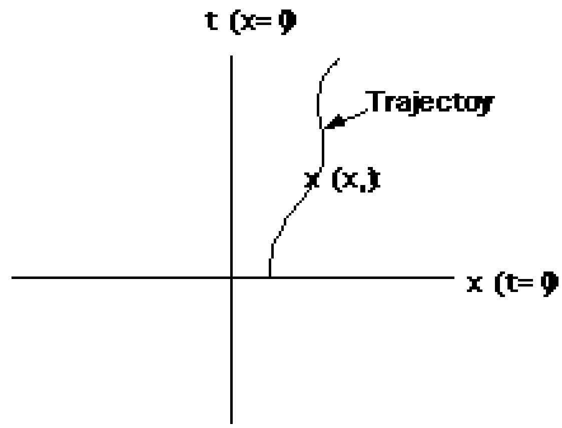

Figure A.14: **Action trajectory Trajectory 2**

better known of as the t axis.

$$x \rightarrow x' = -x \tag{A.19}$$

I am showing this transformation in the active view.

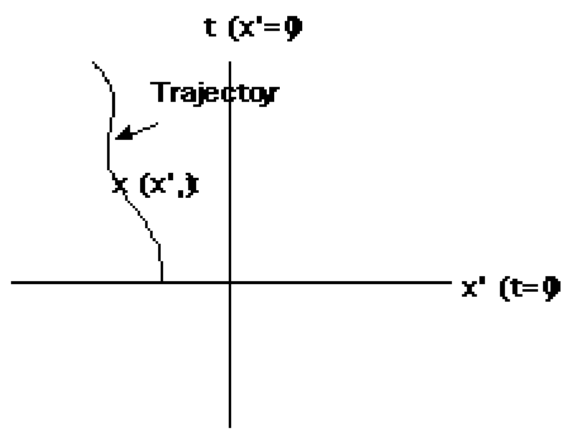

Figure A.15: **Space Reflection** Space Reflection

Space Translation:

Shift the origin of the coordinate system.

$$x \rightarrow x' = x + a \tag{A.20}$$

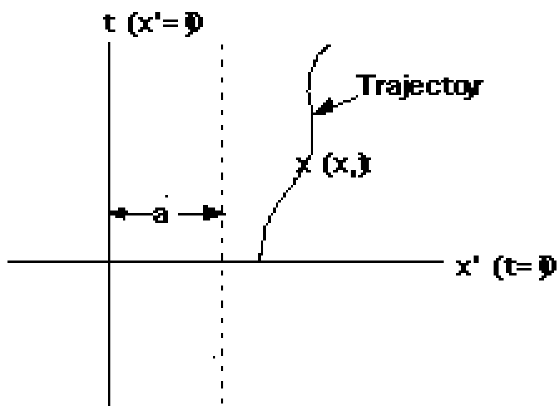

Figure A.16: **Space Translation** Space Translation

Time Translation:

Shift the start of the time.

$$t \rightarrow t' = t + a \tag{A.21}$$

To be a symmetry we will require that the physics before and after the shift is the same. I have not carefully defined what I mean by "the same." I will do so shortly.

Newton's Action at a Distance Law of Gravitation

The law of force that describes the gravitational influence of one body, say body 2, on another body, say body 1, is

$$\vec{F}_{1,2} = G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3} \times (\vec{r}_2 - \vec{r}_1) \tag{A.22}$$

Similarly, the gravitational force of body 1 on body 2 can be found by interchanging the labels of particles 1 and 2.

$$\vec{F}_{2,1} = G\frac{m_2 m_1}{|\vec{r}_1 - \vec{r}_2|^3} \times (\vec{r}_1 - \vec{r}_2) \tag{A.23}$$

Thus if you are operating at the level of the forces you have that if you interchange particles 1 and 2, i. e. change the labels 1 and 2, $1 \leftrightarrow 2$ and get $\vec{F}_{1,2} \rightarrow -\vec{F}_{2,1}$ This is a discrete transformation. If for some reason you are interested in the forces, this is not a symmetry. It is actually a manifestation of the Law of Action Reaction. In other words, we construct the Law of Gravitation so that it obeys the Law of Action Reaction. On the other hand, if you look at the entire set of equations without the forces, there is no change.

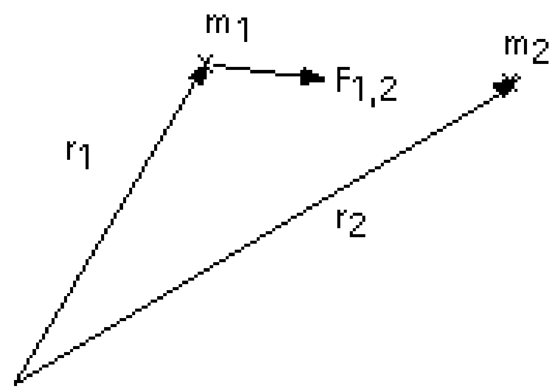

Figure A.17: **Gravitational Symmetry** Gravitational Symmetry

$$m_1\vec{a}_1 = G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3} \times (\vec{r}_2 - \vec{r}_1) \quad (A.24)$$

$$m_2\vec{a}_2 = G\frac{m_2 m_1}{|\vec{r}_1 - \vec{r}_2|^3} \times (\vec{r}_1 - \vec{r}_2) \quad (A.25)$$

Some symmetries of this law:

This is then a symmetry. When you put a shift to all the positions by some amount, $\vec{a}$, nothing changes, i. e. $\vec{r}_i \to \vec{r}_i + \vec{a}$. This is a continuous symmetry. When you replace all the positions with the reverse position, $\vec{r}_i \to -\vec{r}_i$ again nothing changes. Remember $\vec{a}_i \to -\vec{a}_i$. This is a discrete symmetry. If you change all the distances in the problem by a scale $\vec{r}_i \to \vec{r'}_i = \lambda\vec{r}_i$, then this is not a symmetry. But, if you also change the time scale by $t \to t' = \lambda^{\frac{3}{2}}t$, then you have a symmetry. This is a continuous symmetry. Note that the identity transformation is $\lambda = 1$.

A.5 Symmetry and Action

A.5.1 Introduction

You can have the situation that you make the change and the action does not change at all. Said more carefully, you have transformed end points and transformed paths and you get the same value for the action.

Consider the free particle and translations in space.

$$\begin{aligned} x' &= x + a \\ t' &= t \end{aligned} \tag{A.26}$$

This implies that $v' = v$. Thus

$$\begin{aligned} S'(x'_f, t'_f, x'_0, t'_0; path') &= \sum_{path',(x'_0,t'_0)}^{(x'_f,t'_f)} (m\frac{v'^2}{2})\Delta t \\ &= \sum_{path,(x_0,t_0)}^{(x_f,t_f)} (m\frac{v^2}{2})\Delta t \\ &= S(x_f, t_f, x_0, t_0; path) \end{aligned} \tag{A.27}$$

If action is the basis of all physics, then we have a natural definition of a symmetry of a physical system. A physical system has a symmetry if there is a way to modify the system and yet there is no significant change in the action. It is important to be careful about the meaning of significant in this sentence. For most purposes the value of the action is not important. The action primary role is to select a path from the infinity of possibilities. In this sense, we can as a first step assert that the system is symmetric if the system before and after the change still selects the same path as the natural path. You again have to be careful because the same path is actually the same path as seen in the modified system. An example might help clarify this.

Harmonic Oscillator and Symmetry

The harmonic oscillator is one of the most important physical systems. We will discuss the physics of this system in greater detail in a later section, Section **??**, but for now will use it as another example in which to examine the role of symmetry in a physical system. For now just think of of it as a physical system that goes back and forth.

The Lagrangian for the harmonic oscillator is

$$L(v, x) = KE - PE = m\frac{v^2}{2} - k\frac{x^2}{2} \tag{A.28}$$

where k is the spring constant and m is the mass and both are given constants and have the dimension $k \stackrel{\text{dim}}{=} \frac{\text{Mass}}{\text{Time}^2}$ and, of course, m is a mass. Note that, if these are the only two dimensional constants that are available, then you cannot make a length but you can make a time. If you rescale the distances by an amount λ, as follows:

$$\begin{aligned} x &\rightarrow x' = \lambda x \\ t &\rightarrow t' = t \end{aligned} \tag{A.29}$$

which implies that

$$v \rightarrow v' = \frac{\Delta x'}{\Delta t'} = \lambda \frac{\Delta x}{\Delta t} = \lambda v \tag{A.30}$$

The Lagrangian for the new system is

$$L'(v', x') = KE' - PE' = m\frac{v'^2}{2} - k\frac{x'^2}{2} = m\lambda^2(\frac{v^2}{2} - k\frac{x^2}{2}) = \lambda^2 L(v, x) \tag{A.31}$$

So that

$$\begin{aligned} S'_{Path'}(x'_0, t'_0; x'_f, t'_f) &= \sum_{path',(x'_0,t'_0)}^{(x'_f,t'_f)} (m\frac{v'^2}{2} - k\frac{x'^2}{2})\Delta t' \\ &= \lambda^2 \sum_{path,(x_0,t_0)}^{(x_f,t_f)} (m\frac{v^2}{2} - k\frac{x^2}{2})\Delta t \\ &= \lambda^2 S_{Path}(x_0, t_0; x_f, t_f) \end{aligned} \tag{A.32}$$

where Path' is the Path that is at the rescaled distances

$$x'(t') = \lambda x(t) \tag{A.33}$$

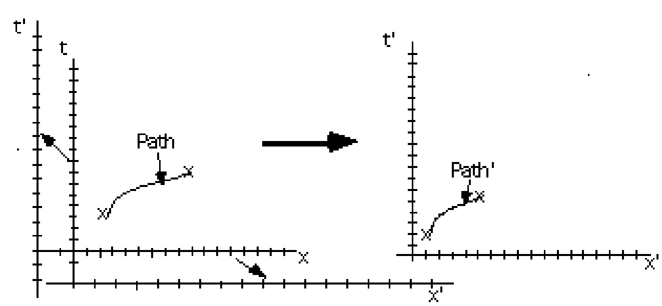

Figure A.18: **Rescale Oscillator** Rescale Oscillator

Path	Action	Path'	Action'
1	S_1	1'	$S'_{1'} = \lambda^2 S_1$
2	S_2	2'	$S'_{2'} = \lambda^2 S_2$
.	.	.	.
.	.	.	.
.	.	.	.
natural	S_{least}	natural'	$S'_{least'} = \lambda^2 S_{least}$
.	.	.	.
.	.	.	.
.	.	.	.

You get the same path even though the calculations are all different.

A.5.2 Galilean invariance

In order to show that the straight line was the solution to the free particle action problem I assumed that the action procedure was Galilean invariant and went to a special frame. The question is "is it." The action is

$$S(x_f, t_f, x_0, t_0; path) = \sum_{path, x_0, t_0}^{x_f, t_f} \Big(m\frac{v^2}{2}\Big)\Delta t \tag{A.34}$$

What happens when you make the Galilean transformation?

$$\begin{aligned} x' &= x - at \\ t' &= t \end{aligned} \tag{A.35}$$

Where a is a parameter that labels the transformations and has the dimensions of a velocity – it is actually interpreted as a velocity. With this transformation all the velocities shift, $v' = v - a$.

$$\begin{aligned} S'(x'_f, t'_f, x'_0, t'_0; path') &= \sum_{path', x'_0, t'_0}^{x'_f, t'_f} \Big(m\frac{v'^2}{2}\Big)\Delta t \\ &= \sum_{path, x_0, t_0}^{x_f, t_f} \Big(m\frac{(v-a)^2}{2}\Big)\Delta t \\ &= \sum_{path, x_0, t_0}^{x_f, t_f} \Big(m\frac{v^2}{2}\Big)\Delta t - \sum_{path, x_0, t_0}^{x_f, t_f} (mva)\Delta t + \sum_{path, x_0, t_0}^{x_f, t_f} \Big(m\frac{a^2}{2}\Big)\Delta t \end{aligned}$$

$$
\begin{aligned}
&= S(x_f, t_f, x_0, t_0; path) - ma \sum_{path,x_0,t_0}^{x_f,t_f} v\Delta t + \left(m\frac{a^2}{2}\right) \sum_{path,x_0,t_0}^{x_f,t_f} \Delta t \\
&= S(x_f, t_f, x_0, t_0; path) - ma(x_f - x_0) + \left(m\frac{a^2}{2}\right)(t_f - t_0)
\end{aligned}
\tag{A.36}
$$

The last two terms are independent of path. Therefore the path selection process the selects the least path in S will select the transformed path in S'. The action changes under the transformation but in an unimportant way. **This is not a symmetry and there is no associated conserved quantity.**When we implement this for special relativity it will become a symmetry.

A.5.3 More on Symmetry and Action

The easiest way to guarantee that the action is symmetric under a set of transformations is to construct it only from the form invariants for that set of transformations. In fact, it is a necessary and sufficient condition that the action is symmetric that it be composed of only form invariants for that set of transformations.

As an example consider the action for a satellite of mass m in orbit around the earth. Locating the earth at the origin, the action is

$$
S(\vec{x}_0, t_0, \vec{x}_f, t_f; path) = \sum_{Path,\vec{x}_0,t_0}^{\vec{x}_f,t_f} (m\frac{\vec{v}^2}{2} + Gm\frac{M_{earth}}{r})\Delta t \tag{A.37}
$$

This action is composed of $\vec{v}^2$ which is a form invariant for rotations about the origin. r is the distance from the origin and it is also a form invariant for rotations. Obviously Δt is a form invariant for rotations. Thus this action has a symmetry that is the set of transformations that are the rotations about the origin.

A.5.4 Noether's Theorem

For every continuous transformation that is connected to the identity that is a symmetry, no important change, there is a conserved quantity. Noether's Theorem also tells you how to construct the conserved quantity. When I tell you what the question is and thus when a change is important, I can tell you how to construct the conserved quantity.

Space translation Symmetry

The conserved quantity that is associated with situations with space translation symmetry is called linear momentum. In certain cases it is $\vec{p} = m\vec{v}$ but not all the time. I will tell you when those cases are.

Rotation translation symmetry

The conserved quantity that is associated with situations with space rotation symmetry is called angular momentum. Rotations are a vector quantity. Again in certain cases it is $\vec{L} = m\vec{r} \times \vec{v}$.

Time translation Symmetry

The conserved quantity that is associated with situations with time translation symmetry is called energy. This is actually the case all the time but the form of the energy may change.

Galilean Invariance

This is almost a symmetry classically and becomes a full blown symmetry in the modern language. First, let's discuss what the transformation is.

There is no experiment that can be performed that can measure the velocity of an moving observer. We can detect the presence of accelerations and measure the relative velocity between two bodies but we cannot measure absolute velocities.

Another way to say the same thing is that, if you are not accelerating, you are always at rest in your own rest frame.

In the language of transformations, all the laws of physics must be invariant under a transformation of the form

$$\begin{aligned} \vec{x} &\rightarrow \vec{x}' = \vec{x} + \vec{R} + \vec{v}t \\ t &\rightarrow t' = t \end{aligned} \tag{A.38}$$

where $\vec{R}$ and $\vec{v}$ are constants that are the parameters that label the continuous transformations. They can be interpreted in terms of two coordinate systems this can be interpreted as the difference in the measurements of two relatively displaced and relatively moving coordinate systems.

Although this is a continuous symmetry that is connected with the identity, it is not a symmetry classically. I will explain this later. Since this is

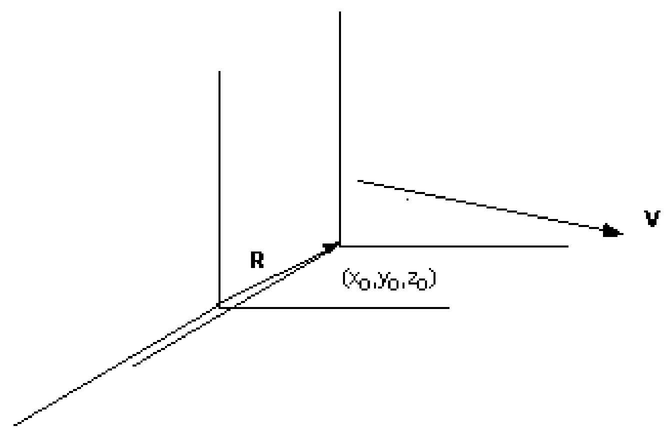

Figure A.19: **Galilean Invariance** Galilean Invariance

not a symmetry, there is no conserved quantity that is the result of Galilean invariance in classical physics.

You should apply this transformation to the gravitational force above and see that the neither the forces nor the equations change. If you use these as your criteria for a symmetry, this would be a symmetry. It is not so we see that we need a better criteria.

Add some notes on the two observers moving by each other.

Please read the Feynman lecture. I do not expect that all of you will follow this material. It is a basis for Noether's Theorem.

Consider a change in the system that also changes the description of initial and final events. This is what will generally happen. Here, when you do the transformations, you will get in addition to the usual terms of the integral of the Lagrangian but also terms from the end points. Our modified form of Feynman's equation

$$\begin{aligned} \delta S &= \left(\left(\frac{\delta L}{\delta v}\mid_{x_{nat}(t)}\right)\delta x\right)_{t_f} - \left(\left(\frac{\delta L}{\delta v}\mid_{x_{nat}(t)}\right)\delta x\right)_{t_0} \\ &\quad + \int_{t_0}^{t_f}\left(\frac{d}{dt}\left(\frac{\delta L}{\delta v}\right) - \frac{\delta L}{\delta x}\right)\delta x dt \end{aligned} \tag{A.39}$$

To get the action to be stationary now we will require that as before the integrand vanish

$$\frac{d}{dt}\left(\frac{\delta L}{\delta v}\right) - \frac{\delta L}{\delta x} = 0 \tag{A.40}$$

but also that the terms from the end points vanish. This part simply selects the natural path. To understand the end points consider an example, the simple translation. In this case δx is simply a number that is added to all points in the path.

$$\delta x(t_f) = \delta x(t_0) = a \tag{A.41}$$

or

$$\left(\left(\frac{\delta L}{\delta v}\mid_{x_{nat}(t)}\right)\delta x\right)_{t_f} - \left(\left(\frac{\delta L}{\delta v}\mid_{x_{nat}(t)}\right)\delta x\right)_{t_0} = \left(\left(\frac{\delta L}{\delta v}\mid_{x_{nat}(t)}\right)_{t_f} - \left(\frac{\delta L}{\delta v}\mid_{x_{nat}(t)}\right)_{t_f}\right) a \tag{A.42}$$

Setting this to zero, yields

$$\left(\frac{\delta L}{\delta v}\mid_{x_{nat}(t)}\right)_{t_f} = \left(\frac{\delta L}{\delta v}\mid_{x_{nat}(t)}\right)_{t_f} \tag{A.43}$$

But $\frac{\delta L}{\delta v}\mid_{x_{nat}(t)}$ is what you would **define** as the momentum. It is the momentum when you use the usual Lagrangian. Thus this is nothing more than the statement that momentum is conserved.

$$p(t_f) = p(t_0) \tag{A.44}$$

This is a special case of a general theorem called Noether's Theorem. Given any transformation that can be connected with the identity transformation, no change, by a continuous parameter. There will always be a conserved quantity. In the above example the transformation is translation. In the limit $a \to 0$ you have no translation and thus no change and the identity transformation. In this case, the conserved quantity is the linear momentum.

Another way of looking at this result is that, once you have selected the natural path and if you include the end point variations, the action is a function of the end points only. If the symmetry transformation changes the end points you have

$$\delta S_{Nat}(x_0, t_0; x_f, t_f) = \frac{\delta S_{Nat}}{\delta x_0}\delta x_0 + \frac{\delta S_{Nat}}{\delta x_f}\delta x_f + \frac{\delta S_{Nat}}{\delta t_0}\delta t_0 + \frac{\delta S_{Nat}}{\delta t_f}\delta t_f \tag{A.45}$$

In the case of translations,

$$\delta x(t_f) = \delta x(t_0) = a \tag{A.46}$$

and all the δt_i are zero.

Thus we get

$$\frac{\delta S}{\delta x_f} = -\frac{\delta S}{\delta x_0} = p = \text{constant} \tag{A.47}$$

An Example

For the free particle,

$$S_{natural} = m\frac{(x_f - x_0)^2}{2(t_f - t_0)} \tag{A.48}$$

$$p = \frac{\delta S}{\delta x_f} = m\frac{(x_f - x_0)}{(t_f - t_0)} = mv \tag{A.49}$$

since v is a constant.

We noted above that the satellite in orbit is a case that is invariant under rotations about the origin. This set of transformations is a continuous set and thus there is a conserved quantity. In this case we call it the angular momentum. The construction of this conserved quantity involves cumbersome notation because it only makes sense in a system with at least two spatial dimensions and thus involves vector notation. In addition, it is computationally difficult to find an expression for the natural path. But note that the free particle Lagrangian is also composed only of form invariants for rotations about the origin. Thus this set of transformations is also a symmetry for this case. The analysis is still cumbersome because of the vector notation. I am aware that you will not be able to reproduce this analysis. All that I ask is that you follow it.

We will work in two spatial dimensions. For this case the action is

$$S(\vec{x}_0, \vec{t}_0; \vec{x}_f, \vec{t}_f) = \sum_{\text{NaturalPath}, \vec{x}_0, \vec{t}_0}^{\vec{x}_f, \vec{t}_f} m\frac{\vec{v}^2}{2}\Delta t \tag{A.50}$$

and as we see is composed of only form invariants not only of translations in space and time but also for rotations. The quantity $\vec{v}^2$ is invariant under rotations.

For the natural path the action is

$$S_{natural} = m\frac{(\vec{x}_f - \vec{x}_0)^2}{2(t_f - t_0)} \tag{A.51}$$

and the change in the action caused by the end point changes are

$$\delta S_{Nat}(\vec{x}_0, t_0; \vec{x}_f, t_f) = \frac{\delta S_{Nat}}{\delta \vec{x}_0} \cdot \delta \vec{x}_0 + \frac{\delta S_{Nat}}{\delta \vec{x}_f} \cdot \delta \vec{x}_f + \frac{\delta S_{Nat}}{\delta t_0} \delta t_0 + \frac{\delta S_{Nat}}{\delta t_f} \delta t_f \tag{A.52}$$

For rotations, δt_0 and δt_f are zero. The $\delta \vec{x}_0$ and $\delta \vec{x}_f$ are the displacements of the end points that result from the rotation. For a rotation through an angle θ, they are

$$\delta \vec{x}_0 \tag{A.53}$$

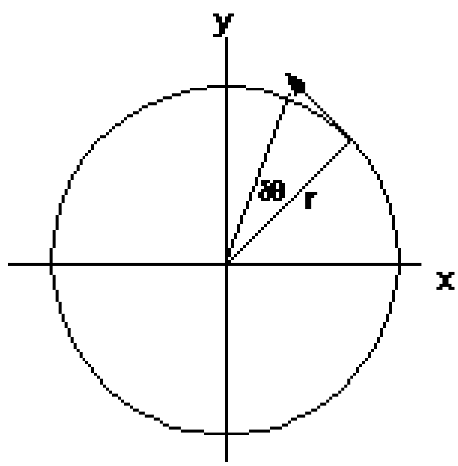

Figure A.20: **Rotation** Rotation.

From the rule above we need the change in the S_{Nat} along this direction.

As in the translation example we see that the change in S with changes in position is the regular momentum. Thus the thing that multiplies $\delta\theta$ in the change in action is the momentum along this direction times the distance. This is what we always called the angular momentum.

Thus we get the rather complicated object

$$L_{axis} = \frac{\delta S_{Nat}}{\delta \vec{x}_0} \cdot r_0 \, (\theta)_0 \tag{A.54}$$

The lesson of all this is that the symmetry implies that there is a conserved quantity. These are the things that we call momenta or energy etc. The form that they take depends on the nature of the Lagrangian.