Chapter 10

Effects of Gravitation

10.1 Linearized Gravity

The theory of gravitation as contained in Einstein's equations, see Section 9.11, is a field theory with the metric as a the local field. Since these are based on the Reimann curvatures which are non-linear in the metric and thus as a field theory in the metric, it is a non-linear field theory. Fortunately, in most applications, the effects of gravity are weak implying that the field itself is weak. More relevantly, most sources of gravity are weak and slow moving. In addition, it is a requirement of the theory that it reproduce Newtonian gravity in the appropriate limit. That limit is characterized is again characterized by the weakness of the gravitational fields and the non-relativistic motion of the sources.

We start with the development of a theory based on an expansion of the metric about a given metric, $g^{(0)}_{\mu\nu}$. This can be any metric but, for most cases such as the Newtonian limit and radiation, the expansion is about the usual flat spacetime Minkowski metric

$$g^{(0)}_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{10.1}$$

Other cases are possible though. For instance for cosmology there may be a non-trivial background metric such as the Robertson-Walker metric, Equation 10.33 or near a strong source such as one that generates a black hole condition the usual Schwartzchild metric, Equation **??**, could be the background. Regardless of the background metric, the linearized theory is an expansion in this section will be to first order in the correction. In order to understand the procedures of the linearized approximation, we will deal here only with the case of a Minkowski background, $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$. The advantage of spacetime independent backgrounds such as $\eta_{\mu\nu}$ will be apparent as the theory is developed. The complications of a more general background will be treated in Appendix D.

The conditions appropriate to the Newtonian limit are then presented and the Newtonian correspondence elaborated. A general class of applications can be analyzed as an analog to the electromagnetic field which is, of course, linear. Finally, the weak field case is applied to the problem of gravitational radiation.

10.1.1 Linearized Theory

Expanding the metric about the Minkowski metric and keeping only first order terms in the correction

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{10.2}$$

As always, $g^{\mu\nu}$ is the inverse metric,

$$g^{\mu\nu} g_{\nu\gamma} = \delta^{\mu}_{\gamma}. \tag{10.3}$$

Since $\eta^{\mu\nu}$ is a metric, it satisfies the same Equation 10.3 which yields $\eta^{\mu\nu} = \eta_{\mu\nu}$. Plugging this into Equation 10.3 and keeping only first order terms,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{10.4}$$

where

$$h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\mu\gamma} h_{\rho\gamma}, \tag{10.5}$$

Note that $h^{\mu\nu}$ is not necessarily the inverse of $h_{\mu\nu}$ since this would be a second order condition on $h_{\mu\nu}$. In other words, we are already treating the perturbation term as a general second rank tensor. Obviously, it is also symmetric.

With this as our metric we begin the construction of the appropriate curvature contributions. Starting with the linearized Christofel symbol

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}\left\{\frac{\partial h_{\sigma\nu}}{\partial x^{\rho}} + \frac{\partial h_{\sigma\rho}}{\partial x^{\nu}} - \frac{\partial h_{\nu\rho}}{\partial x^{\sigma}}\right\}, \tag{10.6}$$

it is straight forward to compute the linearized Riemann mixed tensor since products of Christofel symbols are second order in $h_{\mu\nu}$. Therefore the Riemann tensor is

10.1.2 Analog to Electromagnetism

10.1.3 Gravitational Waves

10.2 Curvature around a Massive Body

10.3 The Geometry and Evolution of the Universe

10.3.1 Background Ideas

After 1916, Einstein and others applied the General Theory of Relativity, the modern theory of gravity to the entire universe. The basic ideas are so simple and compelling that it seems that they must be correct and most of the observational data are in complete concordance. Despite this simplicity, the history of the subject is full of surprising turns and it is worthwhile telling some of this history so that we can understand the context of our current understanding and why this is still an exciting and active research field – hardly a week goes by without some new article in the newspapers indicating some controversial measurement. Like all good science, cosmology is now being driven by new experimental results. It is important to realize that the current controversies in our understanding of the operation of the universe are all really at the interface of General Relativity and micro-physics. In this section, we will deal only with the broadly accepted aspects of the subject and leave the issues that emerge from the interaction of the large scale universe with microphysics to a later chapter, see Chapter 11. Because of this, in this chapter, we will treat the matter in universe very simply and accept forms of matter that are currently not understood.

Einstein had a rather simple outlook on the nature of the universe and its origin. Like Descartes and others before him, he felt that that the universe has always been present or at least reasonably stable. This desire was tempered though by the observation that, although the ages of the sun and planets were quite large, there were certainly dynamical processes taking place in the cosmos. This balance between perpetuity and evolution meant that he wanted solutions for the space-time structure of the universe that had stationary or at least quasi-stationary solutions, i. e. solutions that were stable over long periods of time. We should realize that the astronomy of the period was not nearly as advanced as it is today and the observational situation was that, at all distances, the night sky looked the same. Due to the fact that the speed of light is finite, looking at longer distances was the same as looking back in time. It is just that the distances that we being observed were small compared to what we now know are relevant to cosmological questions. Also, we have been observing the universe seriously for only the last few hundred years and on the lifetime of stars and things like that this is but an instant.

As they were originally proposed the equations for the evolution of space-time, the Einstein equations **??**, did not possess any stationary solutions; there were not enough dimensionful parameters to define a time. He realized that there was a simple way to modify the equations and he added the term now called the cosmological constant.

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} - \lambda g^{\mu\nu} = -8\pi G T^{\mu\nu} \tag{10.7}$$

where λ is the cosmological constant. With this term added, he was able to construct solutions that were stable over long times. Note that the cosmological constant has the same dimensions as the curvature, R, which is an inverse length squared. The equations now have two fundamental dimensional constants.

Two things changed the situation. The great astronomer Hubble observed that the distant galaxies were receding and the the rate of recession was proportional to the distance. We will discuss this observation in more detail in Section 10.3.4 This observation freed Einstein from the illusion that the universe was stationary. In 1922, Avner Freedman produced a set of solutions for the structure of space-time for the universe without the use of the cosmological constant, that were very compelling. In a sense, the Hubble observation allowed Einstein to accept the Freedman solutions as a basis for studies of the structure of the universe. There was another reason that it was easy to accept an expanding universe. Olber predicted that in a stationary universe the night sky should be bright, Section 10.3.3. It is not. Thus with the availability of the Freedman solutions of his equations without the cosmological constant, Einstein dropped the cosmological constant term from his equations and considered his addition of it to them "his greatest mistake."

From the beginning, the Freedman model of the universe was ambiguous about some of the important features of the universe such as its general geometry. Observational data was not only insufficient to resolve these questions, it was also ambiguous. The primary issue centered on whether or not the expansion was slowing down. The acceleration of the universe is hard to observe directly. We have been observing the universe seriously for only a small fraction of its lifetime. The nature of the acceleration of the universe is determined by the energy/matter terms in the Einstein Equation, Equation **??**. The density of matter in the universe is also difficult to measure and what measurements were available were not consistent with the dynamics of galaxies and clusters of galaxies, see Section 10.3.6. Again, through the Einstein Equations, whether or not the expansion was slowing down or speeding up was connected to the question of whether the average curvature was positive or negative. Neither question could be answered.

As would be expected, the Freedman-Hubble expanding universe was not the only candidate for a model of the universe but its theoretical basis was so compelling that it was widely accepted. Not only was the average energy/matter density of the universe important for acceleration, its makeup was determined by the early thermal history of the universe. Speculation on the nature of the matter in the universe was, of course, determined by the micro-physics of the period. Although the nature of the cosmic distribution of matter was difficult to determine observationally, it was clear early on that the matter in the universe was dominantly light nuclei, electrons, and photons. Using information about this mix, it became possible to assign an temperature to the universe. The observation of the 3^0 background radiation in 1964 by Penzias and Wilson which was predicted within the context of a hot Freedman-Hubble model confirmed this family of models but, because it also contained this requirement that the universe be hot, also led to the acceptance of the unfortunate name – Big Bang Cosmology, see Section 10.4.1.

Despite its great success, Big Bang Cosmology had several disturbing features, see Section 10.4.2. Although it is natural to expect that the universe was reasonably homogeneous initially, the observational data was simply too good. There were also predictions from micro-physics of particle species that should have been formed in the early universe and are not observed. Not surprisingly, advances in micro-physics now called the Standard Model, see Section 10.4.4, implied a mechanism for the initiation of the expansion. This is the Inflationary Theory of the early universe, see Section 11.2. More recently, a great deal of observational data of large scale systems has produced more questions and even reopened the question of the role of the cosmological constant, see Section 10.4.3. In fact, the experimental situation with the large scale features of the universe is so compelling that many people are turning to questioning our understanding of the micro-physics that we are using. This is an exciting time to be dealing with cosmological physics. There has emerged a Standard Model of Cosmology that has such a secure observational basis that it is now a serious challenge to the

10.3.2 Copernican Principle

It got Galileo into a great deal of trouble with the church but, today, we have no trouble convincing anyone that the earth is not the center of the universe. Not only that, we have no trouble convincing most people that the sun is also not the center of the universe. You can also get people to accept the idea that the universe is homogeneous, i. e. the laws of physics are the same everywhere. Despite this it is still difficult to convince people that there is no center and no boundary. This is an immediate consequence of homogeneity. It is a fundamental assertion of cosmology that the universe is homogeneous and isotropic. Homogeneous means that all points are the same and isotropic means that at any point all directions are the same. It is easy to think of spaces that are homogeneous and not isotropic, a cylinder. Regardless, if all points are the same, there can be no point being distinguished as a center or a point on an edge.

In a very real sense, this name of Big Bang does not help. Most explosions have a center and certainly all have an edge. This is contrary to our expectations for the universe. Said in a language that we are getting used to, there is no experiment that you can perform that can tell you where you are. Of course, in the cosmological context, this is restricted to very large scales of distance. Here on the earth, we are in a local region that has lots of matter and stuff going on. We can tell where we are and up and down from sideways. The length scale for which the homogeneity holds is one in which the galaxy is a point and even the fact that we are in a small cluster of galaxies is a local density fluctuation that is on a small scale. Note also we are talking about spatial homogeneity. We will discuss what is going on in space-time later when we deal with the evolution of the universe, Section 10.4.

The homogeneity assumption implies that the important physical variables, such as density and so forth, must be independent of position. As stated above, it also means that the laws of physics hold at all places. This is probably the best general test of homogeneity. At large distances, stars work the same way as they do in our galaxy. In addition, all deep sky surveys are consistent with homogeneity at the largest distances. Otherwise, it is hard to make a direct test of homogeneity since we have only occupied this small piece of the cosmos. We are in an awkward situation. We are trying to construct a theory of the universe and we have little experience in it both spatially and temporally.

Isotropy is the statement that at any point all directions are the same. Again, there is no experiment that can differentiate one direction from another. Here we can at least test this hypothesis locally by examining phenomena in all directions. The strongest test of isotropy is the 3^0 background radiation, see Section 10.4.1, which can be tested in all directions. Other than expected small fluctuations, it is shockingly isotropic, maybe too much so, see Section 10.4.2.

Another important test of the homogeneity and isotropy assumption is the pattern of the Hubble Expansion, see Section 10.3.4. The requirement of homogeneity and isotropy restricts the form of the relationship between velocity and distance for remote systems that was observed by Hubble. In fact, the assumptions of homogeneity an isotropy can be used to predict its form uniquely. The fact that it is consistent observationally is verification of these principles.

10.3.3 Olber's Paradox

This was one of the earliest indications that a permanent unchanging universe was not tenable. Basically it is the observation that, in a homogeneous steady universe, the night sky should be bright. Since it is not, there is a problem.

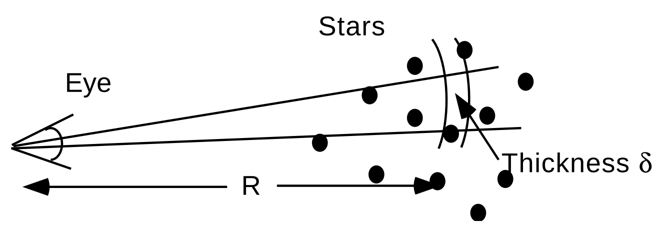

Figure 10.1: **Olber's Paradox** The number of stars that are in a shell of thickness δ in the field of vision at a distance R is proportional to the distance squared. The brightness from each star at the eye falls off as R^-2. In a homogeneous universe, the density of stars is the same everywhere and the brightness is the same. Thus the brightness received at the eye is independent of distance and thus the sky should be bright.

The basis of this prediction is that as you look out at the night sky, since you see into some finite opening angle, the number of stars that are in your field of view from some distance R, grows as R^2, see Figure 10.1. At the same time, the brightness of the light from a star at a distance R falls off with distance as R^{-2}. Therefore, in a homogeneous steady universe, the net light, the number of stars times the brightness per star, received from the stars is independent of the distance. Adding up the contributions from all distances leads to a very large intensity, a bright night sky. Another way to look at it is to realize that in a homogeneous infinite universe along any direction your sight line must ultimately hit a star. This is Olber's Paradox – why is the night sky dark?

Of course, this picture has to be modified in modern times by our realization that the stars that we see are residents in our local galaxy and that, on the large scale, the points of light in the sky are identified not with stars but with galaxies. Substitute the word galaxy for star in the above explanation and you have the modern version of Olber's paradox.

The earlier explanation was that since there was dust or gases in the cosmos, the light falls off faster than R^{-2} and thus we see the dark patches caused by the extra absorption from the intervening material. In a unchanging universe, this explanation will not work. The intervening dust would absorb the light and heat up and glow until its glow balanced the light being absorbed, see Section **??**. Thus, if the universe is infinite and forever, there should not be a dark night sky.

We will get ahead of our story but it is good to understand the modern resolution of Olber's Paradox. The resolution of the paradox is that the universe is expanding and dynamic, see Section refSec:Hubble. When looking out, we are really looking back in time and, at these earlier times, the stars and galaxies have not yet formed. Thus looking between the stars and galaxies, we should see light from the beginning of the universe. In the our model, this is light from a hot dense homogeneous aggregation of matter and radiation. In fact, what we see in the interval between the galaxies is the light from the universe when it was about 300,000 years old. At this time, the universe was a hot sea of matter and mostly photons. The light that comes into the detectors is the light of last scatter off the surface of this hot body. We cannot see any earlier because that light does not stream out. This is very similar to what we see coming from the sun. The interior of the sun is much hotter than the surface but we see light only from the outer most layer which is the surface of last scattering for the light. The light in the hotter interior layers continue to scatter and thus thermalize as they work their way out. Similarly, we see only the surface at 300,000 years because after that the photons are sufficiently soft and the matter so diffuse that they no longer scatter and these are the ones that come into the detectors. All earlier times the deeper photons are still strongly coupled and thus locally thermalize with the hot dense matter. In addition, as the light from the surface of last scatter at age 300,000 years travel to the detectors, the universe has expanded and the light has been red shifted to longer wavelengths. The light thus appears to be from a body that has cooled adiabatically to a very low temperature and is identified as the 3^0 Kelvin background radiation, see Section 10.4.1. Thus in the modern interpretation, there is no paradox. We do not see the glow of an infinity of stars. In stead, we see the a glow that is the young universe which is dynamic.

10.3.4 Hubble Expansion

Originally realized observationally, the Hubble Law was the statement that remote galaxies are moving away from us and that the recession velocity is directly proportional to the distance of the galaxy from us,

$$\vec{v}_{gal} = H\vec{R}. \tag{10.8}$$

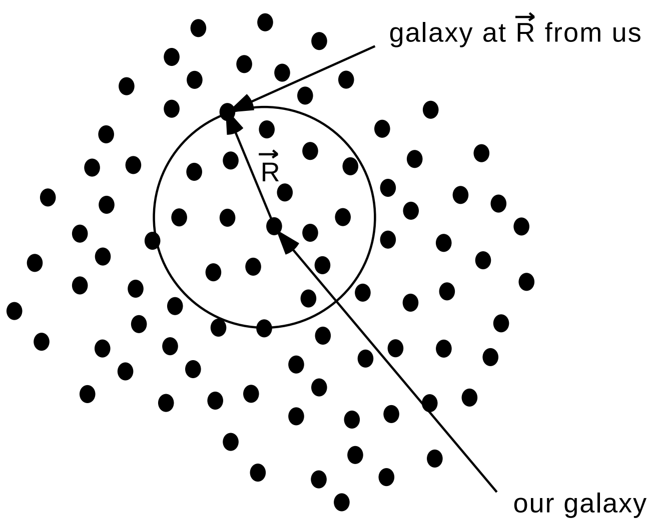

Figure 10.2: **Hubble's Expansion** Hubble's observation that the galaxies are systematically moving away from us. As observed from our galaxy, a galaxy at a relative position $\vec{R}$ from us, has a velocity $\vec{v} = H\vec{R}$. Galaxies at the same distance R have the same speed. The velocity is directed along the relative position away from us. The figure is drawn with our galaxy near the center. You should realize that this is artistic license and does not imply that our galaxy is located in a special part of the universe, see Section 10.3.2.

This simple relationship is the basis of all modern cosmology. The original observations were not very compelling, see Figure **??**. Not only were there few data points but the uncertainty in measuring the distances were rather large. In addition, for nearby galaxies, there may be local motion that distorts the effect. Only on really large distances does the cosmic expansion dominate the velocity. In order to verify this relationship, you need separate measures of distance and velocity. The velocity is actually the easier to measure because of the Doppler shift. The distances are more difficult. Using standard stars, such as variable stars which have a very small range of luminosities, the luminosity can be used to gauge the distance. In fact, now a days, the Hubble Law is now one of the best measures of distance for objects far enough away that the local relative motion is negligible when compared to the cosmic motion. The Hubble plot is a convincing affirmation of the Law, See Figure 10.3.

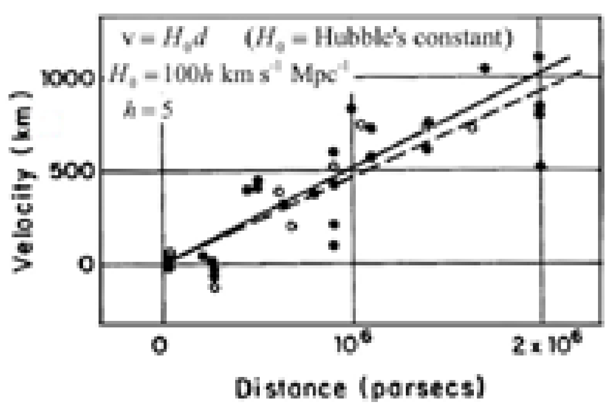

Figure 10.3: **Hubble's Original Plot** The data on the expansion of the universe as presented by Hubble in his original paper in 1929. Subsequent observations have confirmed the conjecture about the expansion of the universe, see Figure **??**.

It takes a great deal of faith to base a theory of the universe on this data but subsequent analysis has confirmed the conjecture.

Insert Current Hubble Plot Here

Note that the Hubble Constant, H, by its definition, is independent of relative displacement, $\vec{R}$, but it can be a function of time. At the time of the laws original formulation, the Hubble Constant was thought to be constant in time but it should be clear that,in any dynamical model of the universe, it will depend on time and in all current models of the universe it does. Of course, if it is a function of time, it is changing at a rate set by the time scale of the universe and, thus, very slowly varying to us. We will thus follow the accepted convention and call it the Hubble Constant despite our anticipation that it varies with time.

A very important fact to note about the Hubble Law is that the Hubble Constant is a scalar; all galaxies at the same distance have the same speed and the direction is along the line of sight from us to the galaxy in question, see Figure 10.2.

This is a strong conformation of the isotropy of the universe. The only directed quantity that enters the law is the relative displacement. There is no directionality coming from the properties of the universe. The universe is acting like a Pascal fluid, see Section 8.5.1. We will take advantage of this fact in preparing simple models of the expansion, see Section 10.4.

In addition, the Hubble Law is an important confirmation of the homogeneity of the universe. For simplicity of argument, consider a one dimensional universe and an expansion pattern that is arbitrary, $v(R)$. Consider the universe and Hubble relationship that would be obtained from a galaxy that is displaced from us by an amount d. Call this galaxies relationship $v_d(R_d)$ where R_d is the distances as measured from that galaxy and v_d is the velocity. In order to have the same physics and thus the same Hubble Law at the original and the new location, $v_d(R_d) = v(R - d)$. That new galaxy of observation is moving away from us at velocity $v(d)$. Thus observations from that galaxy require not only a spatial translation of an amount d but also a Galilean transformation by $v(d)$. Requiring that the relationship between recession velocity and distance be the same for us and the new galaxy is $v(R+d) = v(R) + v(d)$. Requiring this relationship for all R and d implies that $v(R)$ must be linear. Thus we see that homogeneity and Galilean invariance implies the Hubble Law.

We also know that for large relative velocities the Galilean invariance requirement is not simply additive in the velocities and thus must be corrected for the special relativistic effects. Actually, the law is still simply additive in terms of the hangle or relative rapidity, see Section 4.5.

Probably the most significant feature of the Hubble Law is that it provides for the idea of a finite age for the universe. Reverse all the velocities of expansion and the universe compresses into a dense system, ultimately infinite density in a finite time. This is a particularly simple model for the dynamics of the universe but not overly unrealistic. The fact that the Hubble Law provides us with an dimensionful constant that characterizes the universe is enough to infer a finite lifetime for the universe. The dimension of H is t^{-1}. Thus, $\frac{1}{H}$ is a time. As stated earlier, Section 10.3.1, if gravitation is the determining force for the large scale structure of the universe and the universe is homogeneous so that there is only a mass density, there is no time scale in the theory. Thus H^{-1} provides that time scale and, in any reasonable model of the universe, the age of the universe will be of the order of H^{-1}. In fact, when people quote an age for the universe, they are reporting on the latest estimate of H^{-1}. H^{-1} is difficult to measure precisely but observations are settling around a number of the order of 10^{10} years. This is a very satisfying number in the sense that we have not been able to find anything older.

10.3.5 Implications of expansion

For the analysis of this section, we will use non-relativistic physics. This can always work in the sense that we keep the distances and thus the relative velocities small. In addition, we are considering the current epoch of the universe and the energy density is dominated by matter. As a measure of the expansion, we will keep track of the distance to some ring of galaxies which are currently at a distance $R(t)$. Following the Hubble Law this ring of galaxies is moving away from us at a speed HR and these galaxies are gravitationally bound by the sphere of matter/energy contained inside that radius. In the sense of a General Relativistic analysis, we are tracking the expansion in a commoving coordinate system.

Let us examine the energetics of the expansion. The energy of the galaxies at the edge of a sphere of radius R, see Figure 10.2 is the sum of the kinetic and potential energies. For a galaxy of mass m, the potential energy is

$$PE = -\frac{GmM}{R} = -\frac{4\pi R^2 \rho m G}{3} \tag{10.9}$$

where ρ is the mass/energy density of the universe.

The kinetic energy for a galaxy of mass m at this distance is

$$KE = \frac{1}{2} m v^2. \tag{10.10}$$

Using the definition of the Hubble constant, $v = HR$, the

$$KE = \frac{1}{2} m H^2 R^2. \tag{10.11}$$

Thus the total energy of galaxies at the distance R is

$$E(R) \quad = \quad KE + PE$$

$$
\begin{aligned}
&= mR^2\{\frac{1}{2}H^2 - \frac{4}{3}\pi\rho G\} \\
&= \frac{mR^2H^2}{2}\{1 - \frac{8\pi\rho G}{3H^2}\}
\end{aligned} \tag{10.12}
$$

Note that, because of the homogeneity assumption, H and ρ are independent of position. This energy is positive or negative at all R and is the same sign no matter what the value of R. Thus the sign of this energy is a measure that is universal in the universe. We will find later, Equation 10.18, that, if the energy is negative, the galaxies will stop expanding and later start to fall back. Thus if E is positive, the galaxies will continue to expand indefinitely. Thus, there is a critical mass density of the universe that denotes the boundary between continued indefinite expansion and slow down and ultimate collapse.

Using the dimensional content of H and G, we can define a mass/energy density

$$\rho_{crit} \equiv 3\frac{H^2}{8\pi G}. \tag{10.13}$$

Since H is universal this is the critical density everywhere as expected on the basis of homogeneity. Also since $H = \frac{\frac{dR}{dt}}{R}$ where as stated above R is a commoving coordinate, if there is acceleration in the commoving coordinate, H and thus the critical density changes with time.

The energy of a galaxy currently at distance R_N from us is

$$E(R_N) = \frac{mH_N^2R_N^2}{2}\Big(1 - \frac{\rho_N}{\rho_{crit\,N}}\Big), \tag{10.14}$$

where the subscripts N indicate that we are using the current value.

Defining

$$\Omega \equiv \frac{\rho}{\rho_{crit}}, \tag{10.15}$$

where both densities are taken at the same time, this energy is

$$E(R_N) = \frac{mH_N^2R_N^2}{2}\Big(1 - \Omega_N\Big). \tag{10.16}$$

The criteria for the positivity of the expansion energy of the universe in the current epoch is simply whether or not $\Omega_N > 1$.

Equation for evolution of the scale factor

The energy expression, Equation 10.12, can be used to calculate the evolution of $R(t)$. It is interesting to note that we have been calculating a

Newtonian Cosmology. There is no field theory of gravity with finite propagation effects or general or special relativistic corrections. This turns out to be okay because of the judicious choice of the commoving coordinate system. Later we will look at the General Relativistic approach, see Section **??** and compare that approach with this one. The advantage of this Newtonian analysis besides its conceptual simplicity is the references to our usual intuition of dynamics. The three things that we are doing that would not have been appropriate to a true Newtonian cosmology is identifying the evolutionary nature of the universe associated with the cosmological expansion, identifying the space time with the galactic expansion, and using as the source of gravity the mass/energy. In addition, none of the current analysis treats issues of geometry of space let alone space time.

Using Equation 10.12, the energy per unit mass of a galaxy on the shell at R_N is

$$\frac{E(R_N)}{m} = \frac{1}{2}H_N^2 R_N^2 - G\frac{4\pi}{3}\rho_N R_N^2 \tag{10.17}$$

In the same notation, the energy for the galaxies in the same shell at a latter time is

$$\begin{aligned} \frac{E(R(t))}{m} &= \frac{1}{2}\Big(\frac{dR}{dt}\Big)^2 - G\frac{4\pi}{3}\rho_N\frac{R_N^3}{R(t)} \\ &= \frac{1}{2}\Big(\frac{dR}{dt}\Big)^2 - \frac{H_N^2R_N^2}{2}\Omega_N\frac{R_N}{R(t)} \end{aligned} \tag{10.18}$$

where the mass/energy contained within the shell, $M_{inside\,R_N} \equiv \frac{4\pi}{3}\rho_0 R_N^3$, has been conserved.

Equation 10.18 has the same dependence as the one for an object of unit mass being projected to a height, $h = R(t)$, on a body of mass $M_{inside\,R_N}$. Thus if we require conservation of energy for commoving elements for all time, $E(R(t)) = E(R_N)$, then, if $E(R_N)$ is positive, $\frac{dR}{dt}$ will increase indefinitely and, in a sense, escape the massive body. If $E(R(t))$ is negative, the projected body would have slowed and eventually turn around and start to fall back.

For instance, setting $E(R(t)) = E(R_N)$, or, better said the energy of expansion, Equation 10.16, we find that, if Ω_N is greater than one, the greatest distance that a galaxy, which is currently at distance R_N, will be from us is

$$\begin{aligned} R_{max} &= \frac{8\pi G\rho_N R_N}{3H_N^2(\Omega_N - 1)} \\ &= R_N\frac{\Omega_N}{\Omega_N - 1}. \end{aligned} \tag{10.19}$$

Similarly, if $\Omega_N < 1$, $\frac{dR}{dt} > 0$ for all time.

The expansion energy can be used to find the general expression for $\frac{dR}{dt}$,

$$\left(\frac{dR}{dt}\right)^2 = H_N^2 R_N^2 \left(1 - \Omega_N \left(1 - \frac{R_N}{R(t)}\right)\right). \tag{10.20}$$

Since $\frac{dR}{dt} > 0$, the positive root is the appropriate choice.

$$\left(\frac{dR}{dt}\right) = H_N R_N \sqrt{1 - \Omega_N \left(1 - \frac{R_N}{R(t)}\right)}. \tag{10.21}$$

Both for reasons of simplicity and ease of interpretation, it is best to use rescaled variables, the distance in units of R_N, $\alpha \equiv \frac{R(t)}{R_N}$ and times in units of H_N^{-1}, $\tau \equiv H_N t$, Equation 10.21, takes the particularly simple form

$$\frac{d\alpha}{d\tau} = \sqrt{1 - \Omega_N \left(1 - \frac{1}{\alpha}\right)}. \tag{10.22}$$

α is often called the scale factor of the universe.

Two features of this result are important to note. Firstly, we have a one parameter, Ω_N, family of universes. Depending on the value of Ω_N, and only on Ω_N, the universe will either forever expand or reverse expansion and collapse. If $\Omega_N > 1$, the term in the square root is always positive and the system will expand forever. If $\Omega_N < 1$, the term with the square root can vanish and the universe will collapse back onto itself. Secondly, the acceleration is easy to compute,

$$\frac{d^2\alpha}{d\tau^2} = -\frac{\Omega_N}{2\alpha^2}. \tag{10.23}$$

There is no surprize in this result. This is Newton's Law of Gravitation applied to the commoving galaxy in these new variables. In fact, the first integral of this expression is the energy of expansion, Equation 10.12. This acceleration is negative definite. Gravity is the only force operating and it is always attractive. In fact, measurement of a positive acceleration is a special problem for this approach to cosmology. Recent observations indicating the presence of a positive acceleration, [**?**], present a special problem for this approach. We will see that, in the General Relativistic approach, there is the possibility of positive accelerations but that it will require a form of matter that is not consistent with our current understanding of microscopic physics or an uncomfortable value for the cosmological constant, see Section **??**.

In addition, Equation 10.22 is easy to integrate although the closed form solution is not particularly useful. The boundary condition is obviously $\alpha(\tau = now) = 1$. Choosing the origin of time such that $\tau = \text{now} = 1$, we can plot the evolution of the scale factor of for times earlier than now, see Figure 10.4 and in Figure **??** for longer times for three values of the Ω_N; $\Omega_N = 0.5$,$\Omega_N = 1$, and $\Omega_N = 1.5$. In Figure **??**, The universe starts from the time that the scale factor vanishes. It can be seen from Figure 10.4 that the current age of the universe is not strongly dependent on Ω_N and is the order of the inverse Hubble constant as expected. shinola

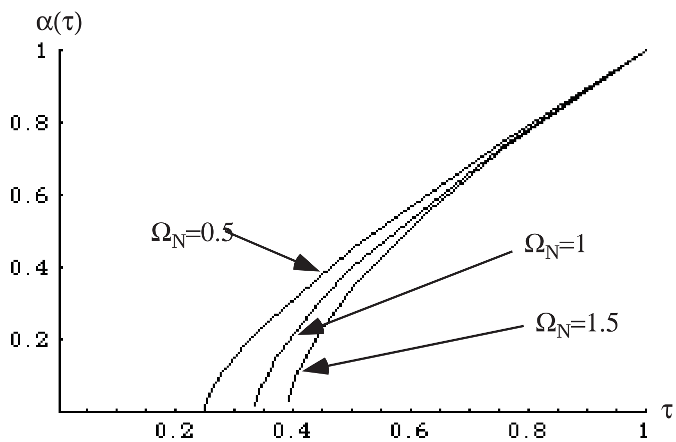

Figure 10.4: **Evolution of the scale factor for early times** The evolution of the scale factor depends only on the mass/energy in the universe. Three cases for the mass/energy density are shown: $\Omega_N = 0.5$ which is an ever expanding universe, $\Omega_N = 0.5$ which is at the transition between collapsing and ever expanding, and $\Omega_N = 1.5$ which is collapsing universe

Evolution of Density

Using the fact that the mass/energy in any commoving shell is conserved, $M_{inside\,R(t)} = M_{inside\,R_N}$, the density scaling law becomes

$$\rho = \frac{\rho_N}{\alpha^3}. \qquad (10.24)$$

Putting this expression into Equation 10.23, the acceleration of the scale factor becomes

$$\frac{d^2\alpha}{d\tau^2} = -\frac{4\pi}{3}\rho\alpha\frac{G}{H_N^2}. \qquad (10.25)$$

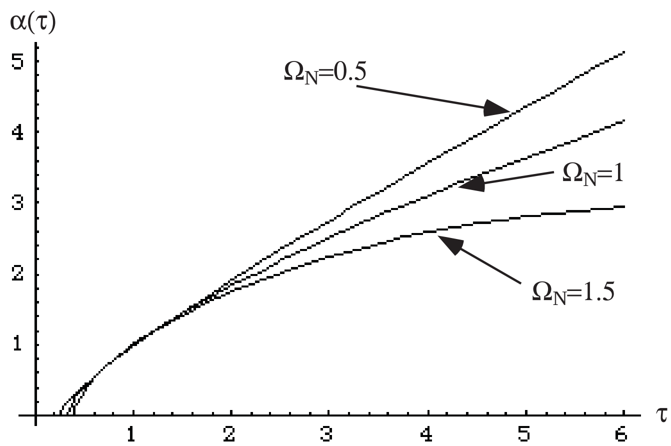

Figure 10.5: **Long time dependence of the scale factor** The evolution of the scale factor depends only on the mass/energy in the universe.

This result shows the Newtonian gravitational basis for the acceleration of the scale factor, it is not as useful as it may appear since we need to find the evolution of the density to integrate it. From the density scaling law, Equation 10.24,

$$\begin{aligned} \frac{d\rho}{d\tau} &= -3\frac{\rho}{\alpha}\frac{d\alpha}{d\tau} && (10.26) \\ &= -3\rho\frac{H}{H_N}. && (10.27) \end{aligned}$$

Again this expression is not as useful as it seems. We require the solution for $H(\tau)$ in order to integrate it.

Similarly, the evolution of the density in terms of α follows from the scaling law, Equation 10.24 and Equation 10.26 as

$$\begin{aligned} \frac{d\rho}{d\tau} &= -3\frac{\rho_N}{\alpha^4}\frac{d\alpha}{d\tau} \\ &= -3\frac{\rho_N}{\alpha^4}\sqrt{1-\Omega_N\left(1-\frac{1}{\alpha}\right)}. && (10.28) \end{aligned}$$

Given the solution of Equation 10.22, this equation can be integrated to give the evolution of the density.

Evolution of H

Given the acceleration of the scale factor, Equation 10.23, it is straight forward to get the equation for the evolution of H

$$\begin{aligned}
\frac{d}{d\tau}\left(\frac{H}{H_N}\right) &= \frac{d}{d\tau}\left(\frac{\frac{d\alpha}{d\tau}}{\alpha}\right) \\
&= \frac{\frac{d^2\alpha}{d\tau^2}}{\alpha} - \frac{\left(\frac{d\alpha}{d\tau}\right)^2}{\alpha^2} \\
&= -\frac{\Omega_N}{2\alpha^3} - \left(\frac{H}{H_N}\right)^2 \\
&= -\frac{1}{\alpha^2}\left(1 - \Omega_N + \frac{3\Omega_N}{2\alpha}\right). \qquad (10.29)
\end{aligned}$$

which is manifestly negative definite as expected.

This model contains all of the large scale features of what is termed the "Big Bang" cosmology. There are features of this model that have not been dealt with such as the nature of the mass/energy in the universe. These will be dealt with later when microphysics has been included. Suffice at this point to say that the matter considered is ordinary matter that obeys all the usual rules of macroscopic and microscopic matter physics such as thermodynamics and our latest discoveries of elementary particle physics. These matters will all be discussed in Section **??**. In addition, there has been no discussion of the space/time geometry. This will require the use of General Relativity which is dealt with in Section 10.4.

One property of the mass/energy that is clearly important is the amount. Ω_N is the only parameter that labels our models of the universe and thus determines whether the universe will expand forever or will eventually fall back on itself and collapse.

10.3.6 Missing Mass

As can be expected, it is very difficult to measure the mass/energy density of the universe. There are several reasons for this. We are not in a region of the universe that is typical. Our planet is in a solar system about a star that is in a galaxy that is a part of a local cluster of galaxies. The star that we orbit is at least a second generation star and thus the matter that is around us is not cosmic in origin. Most significantly, until very recently, the only observable tool was the light from or absorbed by the matter. In fact, all that you can directly observe is the luminous matter. You have to infer the mass from the nature of the light.

Luminous Matter

The standard procedure is to look at the glow of standard objects whose mass can be inferred from other properties of the object. Models of stellar structure provide a tight relationship between the glow of stars and their mass. Galaxies are made of stars and thus we can infer the mass of the glowing material of the galaxies. Thus a ratio of luminosity to mass and assumed proportionality can be established for the mass associated with all the luminous objects observed in the universe and from this a density of matter. In all cases, for the systems in consideration, the mass dominates the mass/energy density. Of course, there could be cool dark objects and often you will hear arguments for their contribution to the mass density of the universe. The occurrence of these kinds of things at a rate sufficient to contribute significantly to the mass density provides theoretical astronomers with lots of speculative freedom and opportunities to publish. It should also be clear that this estimate is at best correct to within a factor of two. The current best estimate is that the mass associated with luminous matter is

$$\Omega_{N_{lum}} \approx 0.01 \tag{10.30}$$

or less.

Gravitational Mass

Besides using the luminous matter, we can infer mass from its gravitational effects. Assuming that the stars in galaxies are gravitationally bound and If you look at the speed of stars then you can estimate the mass that is the source of the gravity that is binding them.

Figure of rotation curves

The mass required to provide dynamic equilibrium is approximately 10 times the luminous mass. This increases the critical density to

$$\Omega_{N_{grav}} \approx 0.1 \tag{10.31}$$

In addition, the galaxies are clustered. We are in group called the local cluster. If you assume that these clusters are not accidental combinations but are also gravitationally bound, there is dark mass between the galaxies. Adding in this mass increases the critical density to

$$\Omega_{N_{clus}} \approx 0.2 \tag{10.32}$$

Einstein had a theoretical prejudice for a universe with $\Omega_N > 1$. We have not yet discussed the space time structure of the universe, see Section 10.4, but in the same way that the values of Ω_N determines the collapse or expansion of the universe, it determines the nature of the geometry. This should be no surprize since a collapse would imply a finite timelike geodesic. In a fully relativistic treatment, a finite timelike world line implies finite spacelike geodesics and, thus, a finite universe. In this case, there is no need for boundary conditions on the universe at its start. Thus, there was a reason to feel that there should be more matter in the universe than that which was observed by the these two methods. This became known as the "missing mass" problem. More recently, there has been a theoretical prejudice for the case $\Omega_N = 1$. This is driven by the need for an inflationary phase at the start of the universe, see Section 11.2. Regardless, there was a strong desire to find more matter than could be seen, luminous, or felt, gravitational. The problem now is that positive accelerations have now been observed and the best description of the large scale structure of the universe, the "Standard Model" , Section 10.4.4 requires dark matter and dark energy. Neither of these seem to be consistent with our current understanding of the nature of matter as developed in microphysics.

10.4 The space time structure of the universe

Before elaborating further on the difficulties with a simple expansion model of the universe, we will redo the analysis of the above section, Section 10.3.5, using the tools of general relativity still restricting ourselves to a simple picture of the nature of the matter in the universe. This will enable us to understand the geometry of the universe and to better understand the role of the dark energy.

Using the arguments of homogeneity and isotropy you can show that the general form of the metric is

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)\{\frac{dr^2}{1-kr^2} + r^2 d^2\Omega\} \tag{10.33}$$

where $R(t)$ is a function of time and is determined by Einstein's equation if you know the energy and momentum densities. $R(t)$ is called the scale factor of the universe. k is a constant that takes on the values 1,0, or -1.

Using this metric you can get all the curvatures. The three space curvature in $\frac{k}{R^2}$. Thus the three space is positively curved for $k = 1$. It is flat if $k = 0$ and negatively curved for $k = -1$. For $k = 1$ the geodesics are all finite in length and thus have finite volume. The other two spaces have infinite geodesics and thus infinite volumes. We can thus identify the three cases that we have here with the values of the critical density that we had above. $\Omega_N > 1$ is the closed positively curved universe. $\Omega_N = 1$ is the case of the flat space and $\Omega_N < 1$ is the negatively curved universe. These last two cases have infinite geodesics.

Whether or not the universe is finite or infinite is determined by the mass density of the universe. It is clear that the value of $\Omega_N$is an important parameter.

10.4.1 Black Body Background

10.4.2 Problems with the Expanding Universe

10.4.3 The Cosmological Constant

10.4.4 The Standard Model of the Universe