Chapter 9

Geometry and Gravitation

9.1 Introduction to Geometry

Geometry is one of the oldest branches of mathematics, competing with number theory for historical primacy. Like all good science, its origins were based in observation and, with historical hindsight, we realize that the evident truths discovered by early geometers were really a result of limited perspective. But like them, for our discussion, we will take certain ideas as evident and as the basis for what we understand. The idea of the point and connected sets of points and particularly the idea of the straight line. As is evident from our discussion of Special Relativity, see Sections 3.3.5 and **??**, we take the straight line to be the shortest distance between two points in space and the longest distance between two events in space-time.

Geometry developed from the need to measure land surfaces for agricultural purposes. The geometry that developed was what we now call plane geometry and the basis for it was first clearly articulated by Euclid and thus the name Euclidean geometry. Euclid set the foundation for plane geometry by means of a set of axioms, evident truths. Modern formulations of geometry realize that there are consistent systems that do not have the same set of axioms. The question then becomes one of choice or appropriateness. In fact, if the early geometers had considered the geometry that is appropriate to large distances on the earth, they would have developed a geometry that was not Euclidean. This alternative geometry is well known and is called spherical geometry. It differs from the Euclidean with the replacement of one axiom, the axiom of parallels. In Euclidean geometry, the axiom of parallels states that given a straight line and a point not on that line that there is one and only one straight line through that point that never touches the the original line no matter how far the lines are extended and that line is called parallel. In spherical geometry, the straight lines are the arcs of great circles, circles on the surface whose center is the center of the sphere. A point to note is that the center of the sphere is not on the surface. In the case of the sphere, all straight lines through a point not on the original line meet the original line, in fact twice. There is a line through a point not on the original line that requires the greatest distance to the nearest intersection of extension before meeting. This line at that point is said to be locally parallel to the original line and this line is unique.

Because in spherical geometry the axiom of parallels is no longer valid, many of the usual rules of Euclidean geometry no longer hold. The sum of the interior angles of a triangle do not add to π but is always greater than π. Think of a triangle on the sphere of the earth formed by the equator and two lines of longitude. At the equator the two lines are locally parallel and the angle between them and the equator is $\frac{\pi}{2}$. They will meet at the north or south pole at some non-zero angle and thus the sum of all three angles is greater than π. Make a square, a four sided figure of equal length sides with all sides meeting at right angles, on the surface. In contrast to the Euclidean case, it does not stop and start at the same point but over-closes, two of the legs of the square meet before the full side length is achieved. A third test is to make a circle, a set of points that are equidistant from some point, on the earth. The ratio $\frac{\text{circumference}}{2r}$, where r is the distance from the point to the circle defined as the radius, is less than π. To most people this is trivial. The problem is that we are measuring on the surface of the sphere. In the underlying three dimensional space in which the sphere is imbedded, the geometry is Euclidean and the world makes sense. For instance, if, instead of the distance as measured from the center on the sphere, the distance used, r', is the distance to the axis that is perpendicular to the plane of the circle passing through the center, the usual result that the ratio $\frac{\text{circumference}}{2r'}$ is π. Because this first identification of a non-Euclidean geometry was on an imbedded sphere, these non-Euclidean geometries are now called curved spaces. This is an unfortunate accident of history as we will discuss shortly but it is so prevalent that everyone uses these terms and we will continue to use this nomenclature. Geometries are flat, Euclidean, or curved, non-Euclidean, with an example being a two dimensional spherical surface imbedded in a flat, Euclidean, three space.

9.2 Gaussian Curvature

The next significant step in the development of modern geometry was taken by the great mathematical physicist Gauss. Gauss was interested in the general problem of the shape of a two dimensional surfaces in our three dimensional space. Instead of a plane, the basis for Euclidean geometry, or a sphere the basis for spherical geometry, consider a two dimensional surface in the shape of a pear imbedded in three space. At a point on the surface there are various curvatures, using an intuitive idea that will be articulated with greater care shortly. At the points near the bottom or top of the pear the surface is much like that of a sphere while in the neck region there is a another type of bend. Also at any point, if the region of examination is small enough, the geometry acts as if it is Euclidean or flat, i. e. for a small enough triangle, the sum of the interior angles of triangles is π.

In order to proceed, Gauss needed a definition of curvature. It had to be local, at a point, and agree with our intuitive notions about curvature. The basic idea is that, on a curved surface, as you move through nearby points on the surface, the normal to the surface changes direction. Thus he produced the following construction: as you move over an element of area on the surface, the tip of the unit normal will paint an area on the unit sphere, see Figure 9.1. the curvature at a point on the surface is the ratio

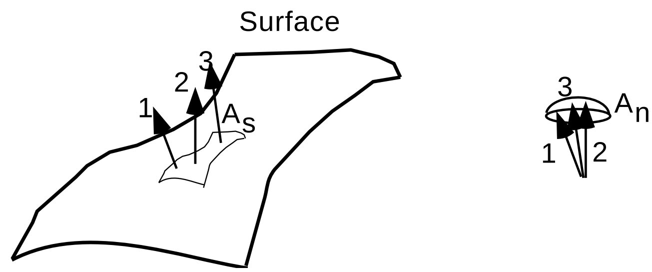

Figure 9.1: **Gauss's Definition of Curvature** Gauss defined curvature as the ratio of the area generated by the tips of the unit normals, $\mathrm{A_n}$, for an element of area, $\mathrm{A_s}$, on the surface as the area on the surface, $\mathrm{A_s}$, goes to zero, $K_\mathrm{G} \equiv \lim_{\mathrm{A_s}\to 0} \frac{\mathrm{A_n}}{\mathrm{A_s}}$.

of the area generated by the tips of the unit normals, $\mathrm{Area_n}$, for an element of area, $\mathrm{Area_s}$, on the surface as the area on the surface goes to zero,

$$K_\mathrm{G} \equiv \lim_{\mathrm{Area_s}\to 0} \frac{\mathrm{Area_n}}{\mathrm{Area_s}}. \tag{9.1}$$

In order to appreciate the subtlety of this construction, let's consider several examples. A flat surface has no curvature since the normal is always the same and thus the $Area_n$ that is generated is that of a point and thus the $Area_n$ is zero. On a sphere of radius r, using the usual spherical coordinates, θ and ϕ, a patch of $Area_s = r^2\delta\theta\delta\phi$ and the normal which is the radius vector generates an $Area_n = \delta\theta\delta\phi$. Thus the curvature is $\frac{1}{r^2}$. This construction shows that this idea of curvature makes sense and that the limit defining it exists for reasonably shaped surfaces. Also note that in the limit of large r the curvature is zero. Now consider a point on the neck of the pear mentioned above. Another example and probably easier to visualize is a Pringle potato chip, see Figure 9.2.

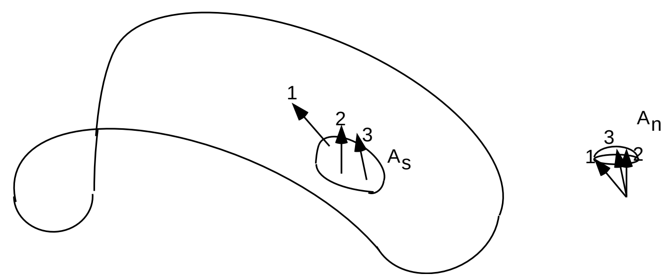

Figure 9.2: **Curvature of a Pringle** A Pringle is an example of a negatively curved surface. The area, A_n, generated by the normals to the surface, A_s, at any point is not zero. The difference between this case and the sphere though is that the area, A_n, is oppositely oriented from that of the area on the surface, A_s, i. e. a right hand coordinate plane on A_s generates a left handed coordinate system on A_n, see Section 9.3.

9.3 Example of negative curvature: the Pringle

I have no idea how Pringles are manufactured, but I will construct my Pringle-like surface by taking a circle of radius R_1 centered on the origin in the two plane, (x, z), displacing it by R_2, $R_2 > R_1$, and then making this circle a surface of revolution about the z axis. This generates a torus or donut shape. We can take a segment of the inner surface, the surface toward the z axis, as our Pringle.

The advantage of this construction is that the labeling of points on the surface and the properties of the normal vector can be determined easily. For example, a point on the surface can be determined from the angle around the original circle as measured from the top most point, θ, and the angle of rotation of the circle around the z axis, ϕ both ranging from zero to 2π.

Using these coordinates, a point on the surface is at

$$\begin{aligned} x &= [R_2 - R_1 \sin\theta] \cos\phi \\ y &= [R_2 - R_1 \sin\theta] \sin\phi \\ z &= R_1 \cos\theta, \end{aligned} \tag{9.2}$$

and the area, A_s, generated by incrementing the two coordinates which are orthogonal is $[R_2 - R_1 \sin\theta] R_1 \delta\theta\delta\phi$. The unit normal vector is along the line from the center of the circle at ϕ and the point on the surface or $\hat{n} = (-\sin\theta\cos\phi)\hat{x} + (-\sin\theta\sin\phi)\hat{y} + \cos\theta\hat{z}$. As the area A_s is swept out, the change in the unit normal is $\delta\hat{n} = (-\cos\theta\cos\phi\delta\theta + \sin\theta\sin\phi\delta\phi)\hat{x} + (-\cos\theta\sin\phi\delta\theta - \sin\theta\cos\phi\delta\phi)\hat{y} + (-\sin\theta\delta\theta)\hat{z}$. Again the lines swept out by the coordinate increments are orthogonal and the area, A_n, generated is $\sin\theta\delta\theta\delta\phi$. The Gaussian curvature is $|K_G| = \frac{\sin\theta}{(R_2 - R_1 \sin\theta) R_1}$.

I have put absolute value signs on this result because the curvature in this case is actually negative. You should realize that, if we choose the coordinate directions in A_s to be right handed in the sense that the normal is outward and generated by rotating directed lines at constant θ into lines of constant ϕ, then the area A_n is left handed in the sense that the image traces of constant θ and ϕ are now left handed. This change in orientation of the areas is the indicator that this curvature is negative and thus

$$K_G = -\frac{\sin\theta}{(R_2 - R_1 \sin\theta) R_1}. \tag{9.3}$$

There are other features of this result that are worth commenting on. The obvious result that the curvature is independent of ϕ is expected. More intriguing is the θ dependence, $K_G(\theta)$. Note that, had we done the analysis for the region $\pi < \theta < 2\pi$, the orientation of the image plane would have been the same as the original element of surface and thus, as given by Equation 9.3, the curvature is positive. At $\theta = \frac{\pi}{2}$, the curvature is $K_G(\frac{\pi}{2}) = \frac{1}{(R_2 - R_1) R_1}$. The square root of the inverse of the curvature is the geometric mean radius of the two circles that make up the surface at this point, the radius of our original circle and the radius of the surface from the axis of symmetry, the z axis. This same observation is also valid for the $\theta = \frac{3\pi}{2}$. This is a general result that we will deal with in more detail in the next section, Section 9.4. The other interesting set of points is at $\theta = 0$ and $\theta = \pi$. Here, the curvature is zero. This can be looked at in two ways. These points are the transition points from the region of negative curvature, the inside of the torus, and the region of positive curvature, the outside of the torus. Since we expect the curvature to smooth, it is required that the curvature vanish at these points. More significantly, This region really is flat in the sense that it is Euclidean.

Think of a cylinder. The curvature of a cylinder is zero – the normal moves along a line as you move around the cylinder but does not change as you move along the axis of the cylinder. Thus, the area, A_n is zero. It is also important to note that the geometry of the cylinder is the same as that of a flat plane; you can unroll the cylinder onto a flat plane. You can do your geometry in the flat plane with the straight lines being the same as usual and the geometry is Euclidean, interior angles of triangles add to π. Thus the cylinder can be covered entirely by a single flat map. You cannot cover a curved surface entirely with a single flat map. You can cover it locally but at some places the distortion caused by the mapping becomes so severe that points are mapped to lines and visa versa. Think of a map of the earth. The usual atlas projection treats the poles, points, as lines. If you exclude the anomalous points by restricting the range of the coordinates you do not cover the earth with a single map but need more than one flat map. This is also a general property of non-Euclidean spaces. Is a cone flat or curved?

9.4 Curvature and Geodesics

In order to proceed further, we will have to examine the general issue of curves in the surface. An arbitrary path connecting two points in the surface can have lots of turns and bends. There are two sources of these, the bends of the surface and the bends of the path within the surface. We can eliminate the bends within the surface by considering only straight line paths between the points. These, by definition, are the shortest distance paths between the points. Since these may be very curved instead of calling them straight lines a better name is geodesic. One of many theorems of the theory of surfaces is that these are unique. These geodesic paths thus contain the bends of the surface and only those bends. In Section 9.5, we will develop a specific differential condition for geodesics that is valid in any coordinate system. For now, we will continue with the more intuitive notions of their properties.

Remembering that our two surface is imbedded in a flat three space, we can identify three directions at any point on the path, the direction along the local tangent to the path, the direction in the surface perpendicular to that direction (Don't forget that, at a point on the surface for a small enough region, the surface is flat and thus this direction is known. To find it, pick another point on the surface not on the original straight line and draw another geodesic through it. These two paths determine a plane, the tangent plane. All geodesics through p share this tangent plane.), and the direction that is perpendicular to these two. This last direction is locally perpendicular to the surface in the sense that the two other directions have generated the tangent plane at the point. This direction is called the normal direction. We already took advantage of these ideas in the identification of the normal to the surface in the previous section, Section 9.2, in which we constructed the Gaussian curvature.

In the neighborhood of the point, the original geodesic is contained in the plane formed from the normal direction and the tangent direction of the geodesic. In the neighborhood of the point p, pick two other points on the original geodesic on opposite sides of p but near p, which will all be in that plane. As is well known from analytic geometry, three points determine a circle. This circle is called the osculating circle. Osculating is from the latin word for "kissing." In some sense, the idea of the osculating circle is the next step up from the tangent. The tangent is determined by two nearby points, determines a magnitude and a direction, and in the limit leads to the concept of the derivative. The osculating circle is determined by three nearby points and utilizes the second derivative, the difference in two tangents, the tangents formed from the original point and the other two points. The inverse of the radius of this osculating circle is called the curvature of the original geodesic. Remember that by using geodesics, there is no bending in the surface. All the bending is due to the surface. There is another geodesic through p that is orthogonal. On that geodesic, construct an osculating circle. Thus at p, for a pair of orthogonal geodesics, there are two osculating circles, one for each of the mutually orthogonal geodesics. As the orientation of this orthogonal pair of geodesics is varied, there will be a direction in which the curvature for each of the orthogonal geodesics will be an extremum. There is no other orientation of the geodesics that have extremum curvatures except trivial variations on this orientation. This last result is called Euler's Theorem. Gauss showed that the Gaussian curvature of the surface as defined in Section 9.2 is the product of these two extremum curvatures,

$$K_G = \frac{1}{R_1 \times R_2}, \tag{9.4}$$

where R_1 and R_2 are the radii of the osculating circles. In addition, the sign of the curvature is determined by the relationship of the two osculating circles. The curvature is positive if both the osculating circles are on the same side of the surface. This is the case for the sphere as discussed earlier.

For the Pringle, Section 9.3, on the inner edge, the osculating circles are on opposite sides of the surface and this is the signature of negative curvature.

As is always the case with the Gaussian curvature, this curvature is an basic property of the surface and does not depend on the coordinate system that we used to make the construction. Granted that the construction of the curvature is most readily done in a coordinate system that is based on a system of orthogonal geodesics, it is still clear from the nature of the Gauss map and Equation 9.10 that the coordinates make the construction possible by staking out the grid but that the local value of the curvature is the same regardless of the coordinate system used. In fact the coordinate system that was used for the torus, Equation 9.2, are not geodesic coordinates; the lines of constant ϕ are geodesics but the lines of constant θ are not. This issue will be discussed in much greater detail later, see Sections 9.5, 9.6, 9.7 and Appendix C.

9.5 The Theorema Egregium and the Line Element

As is clear from Section 9.4, Gauss made an extensive study of the nature of surfaces imbedded in a Euclidean three space. He is responsible for many of the insights and theorems that govern understanding of these surfaces. He was, of course, interested in two surfaces imbedded into the larger three space. He recognized the important role of curvature in defining the nature of the surface; to within an orientation and a translation, the surface is determined by its curvature. His most famous theorem in the theory of surfaces was so striking to him that when he recognized its implications he gave it the title of the Theorema Egregium. A direct translation of the latin would call this the egregius theorem. The modern sense of egregius: outstandingly bad is not the original meaning. The original use of the word was in the sense of outstandingly good and is what is intended in the latin. It was later usage that lead to the current interpretation of egregious as outstandingly bad, see [OED 1971]. It seems that modern young people are not the first ones to reverse the meaning of bad and good when describing things. Regardless, the point of Gauss' name for the theorem was in the sense of outstandingly good. Maybe a better translation would be the Extraordinary Theorem.

This theorem proved that all the important properties of the surface could be developed from information that is intrinsic to the surface and did not need to use properties that were determined by the imbedding of the surface in a Euclidean three space or the coordinate system that was used to do the construction of the Gauss map. The only element that is needed to construct curvature is the length of the line element in whatever coordinate system is being used. In other words, if when you begin to label points on the surface with some set of coordinate labels and, if at the same time, you determined the actual lengths separating nearby coordinate points, you would have all the information that you need to determine the curvature. The other amazing fact is the realization of Riemann that these techniques developed by Gauss carry over to manifolds of any number of dimensions, Section 9.7. The theorem's proof is rather tedious and not really enlightening except in its use of intermediate elements that are very important in our later study of geometry in higher dimensions and thus is worth the effort here for providing and intuitive understanding of the nature of these seemingly abstract quantities.

9.5.1 Proof of Theorema Egregium

Because he was studying two surfaces imbedded in a Euclidean three space, Gauss knew immediately the distances between nearby coordinate labeled points. Given a two surface in the usual three space with a coordinate system, (q^1, q^2), on it, the distance between nearby coordinate labeled points is

$$dl^2 = \Delta\vec{x} \cdot \Delta\vec{x} = \sum_{i,j=1}^{2} \frac{\partial \vec{x}}{\partial q^i} \cdot \frac{\partial \vec{x}}{\partial q^j} \delta q^i \delta q^j. \tag{9.5}$$

This form is called the line element in that coordinate system. The element, $\frac{\partial \vec{x}}{\partial q^i} \cdot \frac{\partial \vec{x}}{\partial q^j}$ can be written

$$g_{ij}(q^1, q^2) \equiv \frac{\partial \vec{x}}{\partial q^i} \cdot \frac{\partial \vec{x}}{\partial q^j} \tag{9.6}$$

is called the first fundamental form or more commonly the metric. This notation is consistent with the Einstein convention, see Section 4.6, and is actually a second rank tensor as indicated, see Appendix C.

We had earlier examples of metrics in coordinates systems in space-time in Section 6.5. Basically, the Theorema Egregium is the statement that the metric is all that is needed for the construction of the curvature at any point on the surface. In other words, we will construct the Gauss map for any point using only the metric and its derivatives. The line element now appears as

$$dl^2 = g_{ij}(q^1, q^2)\Delta q^i \Delta q^j \tag{9.7}$$

where I have also introduced a common notation called the Einstein convention. In any term in an equation, when an index is repeated lower on one variable and upper on another, it is summed over the range of values for that index. There are deeper concepts implied by this notation as explained in Appendix C but for now we are merely using it just as a summation convention.

As a step in proving the theorem, note that $\vec{x}_1 \equiv \frac{\partial \vec{x}}{\partial q^1}$ and $\vec{x}_2 \equiv \frac{\partial \vec{x}}{\partial q^2}$ are tangents to the surface at the point labeled (q^1, q^2). The vectors $\vec{x}_1$ and $\vec{x}_2$ can be used to form a basis in the tangent plane at that point. These vectors and the normal,

$$\vec{n} = \frac{\vec{x}_1 \times \vec{x}_2}{|\vec{x}_1 \times \vec{x}_2|}, \tag{9.8}$$

at that point can be used as a basis in the three space for vectors at that point.

In order to construct the Gauss map, we will need the change in the normal as the coordinate value increases. Define $\vec{n}_1 \equiv \frac{\partial \vec{n}}{\partial q^1}$ and similarly for $\vec{n}_2$. Since $\vec{n}$ is of fixed length,

$$\vec{n}^2 = 1,$$

these vectors lie in the tangent plane

$$\vec{n}_i \cdot \vec{n} = 0.$$

These vectors can now be expanded in the tangent plane using the $\vec{x}_i$ basis,

$$\vec{n}_i = -b_i^j \vec{x}_j. \tag{9.9}$$

The sign convention used here is for convenience later.

In addition for small advances in the coordinate, these vectors generate the area of the Gauss map via

$$Area_n = |\vec{n}_1 \times \vec{n}_2|\, \delta q^1 \delta q^2. \tag{9.10}$$

Since the area of a patch for a small advance in the coordinates is $Area_s = |\vec{x}_1 \times \vec{x}_2|\, \delta q^1 \delta q^2$, the Gaussian curvature is

$$\left|K_G\left(q^1, q^2\right)\right| = \lim_{\delta q^1, \delta q^2 \to 0} \frac{|\vec{n}_1 \times \vec{n}_2|\, \delta q^1 \delta q^2}{|\vec{x}_1 \times \vec{x}_2|\, \delta q^1 \delta q^2} = \frac{|\vec{n}_1 \times \vec{n}_2|}{|\vec{x}_1 \times \vec{x}_2|} \tag{9.11}$$

The sign of the curvature can be recovered from these cross products when it is realized that each of the cross products is a vector in the Euclidean three space and in the space of the Gauss map. If the orientation of the coordinate advance is maintained in both spaces and the vectors of the cross product are the compared the sign of the curvature can be determined. Using Equation 9.9,

$$\begin{aligned} \vec{n}_1 \times \vec{n}_2 &= \left(-b_1^i \vec{x}_i\right) \times \left(-b_2^j \vec{x}_j\right) \\ &= b_1^i b_2^j \left(1-\delta_{ij}\right)(-1)^j \, \vec{x}_1 \times \vec{x}_2 \\ &= \left(b_1^1 b_2^2 - b_1^2 b_2^1\right) \vec{x}_1 \times \vec{x}_2 \\ &= det\,(b)\, \vec{x}_1 \times \vec{x}_2 \end{aligned} \tag{9.12}$$

where

$$\vec{x_i} \times \vec{x_j} = (1-\delta_{ij})(-1)^j \vec{x_1} \times \vec{x_2} \tag{9.13}$$

and

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{9.14}$$

is the usual Kronicker delta and the set of elements b_i^j are treated as a matrix. The factor $det\,(b)$ carries the information of the relative orientation of the cross products. Thus,

$$K_G\left(q^1, q^2\right) = det(b) \tag{9.15}$$

Define $\vec{x}_{ij} \equiv \frac{\partial^2 \vec{x}}{\partial q_i \partial q_j}$. This vector can be expanded in the three vector basis defined by $\vec{x}_1$, $\vec{x}_2$, and $\vec{n}$ at (q^1, q^2),

$$\vec{x}_{jk} = \Gamma^i_{jk} \vec{x}_i + \beta_{jk} \vec{n}. \tag{9.16}$$

It is worthwhile to point out that the expansion coefficients have indices that are meant to be consistent with our summation convention; they may not be tensors of the indicated rank, see Appendix C. Since $\vec{x}_{jk} = \vec{x}_{kj}$,

$$\Gamma^i_{jk} = \Gamma^i_{kj} \tag{9.17}$$

and

$$\beta_{jk} = \beta_{kj}. \tag{9.18}$$

The first coefficient of the expansion, Γ^i_{jk}, is called the Christoffel symbol of the second kind or connection. It is simply related to the Christoffel symbol of the first kind, $\Gamma_{jk,l}$ since

$$\Gamma_{jk,l} \equiv \vec{x}_{jk} \cdot \vec{x}_l = \left(\Gamma^i_{jk} \vec{x}_i + \beta_{jk} \vec{n}\right) \cdot \vec{x}_l = \Gamma^i_{jk} g_{il}. \tag{9.19}$$

The latter term,

$$\beta_{ij} \equiv \frac{\partial^2 \vec{x}}{\partial q_i \partial q_j} \cdot \hat{n} \tag{9.20}$$

is called the second fundamental form. It is an easy matter to connect the elements of the second fundamental form, β_{jk} to the expansion coefficients b^i_j in Equation 9.9 by differentiating $\vec{x}_j \cdot \vec{n} = 0$.

$$\begin{aligned} 0 &= \frac{\partial \left(\vec{x}_j \cdot \vec{n}\right)}{\partial q^k} \\ &= \vec{x}_{jk} \cdot \vec{n} + \vec{x}_j \cdot \vec{n}_k \\ &= \beta_{jk} + \vec{x}_j \cdot \left(-b^i_k \vec{x}_i\right) \\ &= \beta_{jk} - b^i_k g_{ji} \end{aligned}$$

or

$$\beta_{jk} = b^i_k g_{ij}. \tag{9.21}$$

Treating all the elements as matrices,

$$\boldsymbol{\beta} \equiv \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} b^1_1 & b^1_2 \\ b^2_1 & b^2_2 \end{pmatrix} = \boldsymbol{Gb},$$

where the respective matrix definitions are obvious.

Thus taking the determinant of both sides, $det\left(\boldsymbol{\beta}\right) = det\left(\boldsymbol{G}\right) det\left(\boldsymbol{b}\right)$ or, using Equation 9.15,

$$K_G\left(q^1, q^2\right) = \frac{det\left(\boldsymbol{\beta}\right)}{g}. \tag{9.22}$$

where $g \equiv det\left(\boldsymbol{G}\right)$.

Defining $\vec{x}_{ijk} \equiv \frac{\partial \vec{x}_{jk}}{\partial q^i}$,

$$\vec{x}_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial q^l} \vec{x}_i + \Gamma^i_{jk} \vec{x}_{il} + \frac{\partial \beta_{jk}}{\partial q^l} \vec{n} + \beta_{jk} \vec{n}_l.$$

The $\vec{x}_{ijk}$ can be expanded in terms of our vector basis at $\left(q^1, q^2\right)$. Using Equations 9.9 in the fourth term and Equation 9.16 in the second,

$$\begin{aligned} \vec{x}_{jkl} &= \frac{\partial \Gamma^i_{jk}}{\partial q^l} \vec{x}_i + \Gamma^p_{jk} \left(\Gamma^i_{pl} \vec{x}_i + \beta_{pl} \vec{n}\right) + \frac{\partial \beta_{jk}}{\partial q^l} \vec{n} - \beta_{jk} b^i_l \vec{x}_i \\ &= \left(\frac{\partial \Gamma^i_{jk}}{\partial q^l} + \Gamma^p_{jk} \Gamma^i_{pl} - \beta_{jk} b^i_l\right) \vec{x}_i + \left(\frac{\partial \beta_{jk}}{\partial q^l} + \Gamma^p_{jk} \beta_{pl}\right) \vec{n}. \end{aligned} \tag{9.23}$$

Since the order of the derivatives is irrelevant and we get a similar expression for $\vec{x}_{jlk}$, we can construct the null vector

$$\begin{aligned}\vec{0} &= \vec{x}_{jkl} - \vec{x}_{jlk} \\ &= \left(\frac{\partial \Gamma^i_{jk}}{\partial q^l} - \frac{\partial \Gamma^i_{jl}}{\partial q^k} + \Gamma^p_{jk}\Gamma^i_{pl} - \Gamma^p_{jl}\Gamma^i_{pk} - \beta_{jk}b^i_l + \beta_{jl}b^i_k\right)\vec{x}_i \\ &\quad + \left(\frac{\partial \beta_{jk}}{\partial q^l} - \frac{\partial \beta_{jl}}{\partial q^k} + \Gamma^p_{jk}\beta_{pl} - \Gamma^p_{jl}\beta_{pk}\right)\vec{n}.\end{aligned}$$

Since the $\vec{x}_i$ and $\vec{n}$ form a basis, the coefficient of each term must be zero. Defining the mixed Riemann curvature as

$$R^i_{jkl} \equiv \frac{\partial \Gamma^i_{jl}}{\partial q^k} - \frac{\partial \Gamma^i_{jk}}{\partial q^l} + \Gamma^p_{jl}\Gamma^i_{pk} - \Gamma^p_{jk}\Gamma^i_{pl}, \tag{9.24}$$

we have

$$R^i_{jkl} = \beta_{jl}b^i_k - \beta_{jk}b^i_l. \tag{9.25}$$

The Riemann curvature is defined as

$$R_{hjkl} \equiv g_{ih}R^i_{jkl}, \tag{9.26}$$

and therefore satisfies

$$R_{hjkl} = g_{ih}\left(\beta_{jl}b^i_k - \beta_{jk}b^i_l\right) = \beta_{jl}\beta_{kh} - \beta_{jk}\beta_{lh}. \tag{9.27}$$

Thus a particular component of the Riemann curvature is related to the curvature since

$$R_{1212} = det\left(\boldsymbol{\beta}\right), \tag{9.28}$$

or

$$K_G\left(q^1, q^2\right) = \frac{R_{1212}}{g}. \tag{9.29}$$

It may seem strange that one particular component of the 2^4 components plays such a special role. This happens in the special case of a two dimensional manifold. I am now using a the more sophisticated title for a space. A manifold is a space that is defined in such a way that the usual rules for calculus can be applied. Obviously, we want to deal with manifolds. The Riemann curvature enjoys a great deal of symmetry under interchange of its indices. These will be discussed further in Section 9.7. Suffice it to say here that in two dimensions, there is only one independent Riemann curvature and that all the other can be written in terms of that one, chosen to be R_{1212} as

$$\begin{aligned} R_{1212} &= -R_{2112} = -R_{1221} = R_{2121} \\ R_{1111} &= R_{1122} = R_{2211} = R_{2222} = 0 \end{aligned} \tag{9.30}$$

or

$$R_{hjkl} = \left(g_{hk}g_{jl} - g_{hl}g_{jk}\right)\frac{R_{1212}}{g}. \tag{9.31}$$

Despite the great progress in finding new and interesting ways to express the curvature, we have not delivered on our original promise to show that the construction of the curvature can be done intrinsically; that is by using only the metric and its derivatives. First, the relationship between the Christoffel symbol of the first kind and the Christoffel symbol of the second kind, Equation 9.19, can be expressed as a matrix equation,

$$\boldsymbol{\Gamma_j} \equiv \begin{pmatrix} \Gamma_{j1,1} & \Gamma_{j1,2} \\ \Gamma_{j2,1} & \Gamma_{j2,2} \end{pmatrix} = \begin{pmatrix} \Gamma^1_{j1} & \Gamma^2_{j1} \\ \Gamma^1_{j2} & \Gamma^2_{j2} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \boldsymbol{= \tilde{\Gamma}_j G.}$$

This equation can be solved by using the inverse matrix for the metric.

$$\boldsymbol{\tilde{\Gamma}_j {=} \Gamma_j G^{-1}}.$$

The inverse metric matrix can be written in our summation convention notation since

$$\begin{aligned} \boldsymbol{GG^{-1}} &= \mathbf{1} \\ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ g_{ik}g^{kj} &= \delta_i^j \end{aligned} \tag{9.32}$$

or since the inverse of a two by two matrix is well known

$$\boldsymbol{G^{-1}} \equiv \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \frac{1}{g} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix}. \tag{9.33}$$

In the index notation, the Christoffel symbols are related as

$$\Gamma^l_{jk} = g^{li}\Gamma_{jk,i}. \tag{9.34}$$

Taking the derivative of the metric,

$$\begin{aligned} \frac{\partial g_{ik}}{\partial q^j} &= \frac{\partial\left(\vec{x}_i \cdot \vec{x}_k\right)}{\partial q^j} \\ &= \vec{x}_{ij} \cdot \vec{x}_k + \vec{x}_i \cdot \vec{x}_{kj} \\ &= \Gamma_{ij,k} + \Gamma_{jk,i}. \end{aligned}$$

Similarly, calculating $\frac{\partial g_{jk}}{\partial q^i}$ and $-\frac{\partial g_{ij}}{\partial q^k}$ and adding the results,

$$\Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right). \tag{9.35}$$

With the Christoffel symbols in terms of the derivatives of the metric, we have completed the proof of the Theorema Egregium.

In addition, an important feature of the geometry of surfaces is the nature of the shortest lines or geodesics. The idea of the shortest line or "straight line" is an intuitive one and can be carried over to lines on a two surface. Again, if the two surface is imbedded in a Euclidean three space, it is easy to think of the line that bends only as much as required to remain in the two surface and that is the straightest line. That this line is also the shortest distance between two points is again intuitively satisfying. This intuition is reinforced by the realization that a curve in three space can be generated by the intersection of two surfaces. If in the case of our surface of interest we have two points and we want of find the straightest line between them, we can chose from the infinity of planes passing through the two points the one that produces the shortest length line. This line will also be the straightest. The best known example of this is the "straight lines" on the surface of the earth, see Section 9.6.2. In this case, these lines are the intersection of the sphere and the plane that passes through the two points and the center of the sphere, the great circle path. This construction is clearly and extrinsic construction and, if these were the only techniques at hand, we would always have to deal with spaces only after imbedding them. Again, to reduce this problem to an analysis of the intrinsic properties of the space, we will need a procedure that deals with the metric and its derivatives. The geodesic is defined as the curve in the surface between the points that has the least value for the integral of the line element over that curve. As we will see in the examples, Section 9.6, this definition is more restrictive than the intuitive one of straight line above but the geodesic curves are those that bend the least.

Putting this idea into practice, define the curve dependent path length between two points as

$$S\left(q_0^1, q_0^2; q_f^1, q_f^2\right)[path] \equiv \int_{q_0^1, q_0^2; path}^{q_f^1, q_f^2} ds \tag{9.36}$$

where ds is line element

$$ds = \sqrt{g_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds}}ds, \tag{9.37}$$

and the path is a set of parametric functions, $\left(q^1(s), q^2(s)\right)$, of the path length, s, which requires that

$$1 = g_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds}. \tag{9.38}$$

Equations 9.37 and 9.38, look as if they are trivial but when Equation 9.37 is substituted into Equation 9.36,

$$S\left(q_0^1, q_0^2; q_f^1, q_f^2\right)[path] \equiv \int_{q_0^1, q_0^2; path}^{q_f^1, q_f^2} \sqrt{g_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds}}\,ds \tag{9.39}$$

and s is interpreted as a time, this would be the action for a system with two degrees of and a Lagrangian of the form $\sqrt{g_{ij}\frac{dq^i}{dt}\frac{dq^j}{dt}}$. Then we can use the analysis of Section A.2 to minimize the length. The two equations, the Euler-Lagrange equations, Equation **??**, that result from the variation of the functions $q^1(s)$ and $q^2(s)$ are

$$\frac{d}{ds}\left(\frac{\delta}{\delta\left(\frac{dq^l}{ds}\right)}\left(\sqrt{g_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds}}\right)\right) - \frac{\delta}{\delta q^l}\left(\sqrt{g_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds}}\right) = 0 \tag{9.40}$$

for $l = 1, 2$. These equations must be supplemented by the condition that s be the path length, Equation 9.38. This result accomplishes our goal of expressing all the important geometric issues intrinsically, in terms of g_{ij} and its derivatives.

These geodesic equations can be cast into a more geometric form. Carrying out the variational derivatives in Equation 9.40,

$$\frac{d}{ds}\left\{\frac{\frac{1}{2}2g_{lj}\frac{dq^j}{ds}}{\sqrt{g_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds}}}\right\} - \frac{\frac{1}{2}\frac{\partial g_{ij}}{\partial q^l}\frac{dq^i}{ds}\frac{dq^j}{ds}}{\sqrt{g_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds}}} = 0.$$

Expanding the total s derivative and using the fact that Equation 9.38 implies that $\frac{d}{ds}\left(g_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds}\right) = 0$,

$$\frac{1}{\sqrt{g_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds}}}\left\{\frac{\partial g_{lj}}{\partial q^m}\frac{dq^m}{ds}\frac{dq^j}{ds} + g_{lj}\frac{d^2q^j}{ds^2}\right\} - \frac{\frac{1}{2}\frac{\partial g_{ij}}{\partial q^l}\frac{dq^i}{ds}\frac{dq^j}{ds}}{\sqrt{g_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds}}} = 0.$$

or

$$0 = g_{lj}\frac{d^2q^j}{ds^2} + \frac{\partial g_{lj}}{\partial q^m}\frac{dq^m}{ds}\frac{dq^j}{ds} - \frac{1}{2}\frac{\partial g_{ij}}{\partial q^l}\frac{dq^i}{ds}\frac{dq^j}{ds}$$

$$\begin{aligned} &= g_{lj}\frac{d^2q^j}{ds^2} + \frac{1}{2}\left\{\frac{\partial g_{il}}{\partial q^m} + \frac{\partial g_{lj}}{\partial q^m} - \frac{\partial g_{ij}}{\partial q^l}\right\}\frac{dq^i}{ds}\frac{dq^j}{ds} \\ &= g_{lj}\frac{d^2q^j}{ds^2} + \Gamma_{ij,l}\frac{dq^i}{ds}\frac{dq^j}{ds}, \end{aligned} \tag{9.41}$$

where I have relabeled the summed indexes. Using the inverse metric,

$$\frac{d^2q^l}{ds^2} + \Gamma^l_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds} = 0 \tag{9.42}$$

which is the geodesic equation using the connection or Christoffel symbol. The important role of the connection in association with derivatives will be clarified in Section 9.7 and in Appendix C.

Two further points to note. Although we are now in a position to be able to find the curvature and geodesics from the line element in a given coordinate system, does this knowledge tell us all that we need to know about the surface and, clearly, the choice of coordinates has to be irrelevant to these larger questions of what is important to know about the surface. Consider a surface imbedded in a Euclidean three space with a coordinate mesh defined. Any rigid shift in the position or rigid rotation of the surface in the three space does not produce an important change. Since both the first fundamental form and second fundamental form are defined in terms of derivatives of the inner products of the three space coordinates, these are unchanged by these rigid body shifts. Thus, the curvature is not affected by these changes and, in the sense that rigidly rotated and/or shifted surfaces are the same, the curvature is the determinant of the shape of the surface.

The problem of the coordinate dependence of intrinsic knowledge of the surface is not as simple. Two coordinate systems require that the same point in three space have two valid designations, $\vec{x}\left(q^1, q^2\right) = \vec{x}\left(q'^1, q'^2\right)$. Consistency requires that there is a pair of functions, $q'^1 = q'^1\left(q^1, q^2\right)$ and $q'^2 = q'^2\left(q^1, q^2\right)$. In either coordinate system the line element is the same,

$$\begin{aligned} ds^2 &= g_{ij}dq^i dq^j \\ &= g'_{ij}dq'^i dq'^j \\ &= g'_{ij}\frac{\partial q'^i}{\partial q^l}\frac{\partial q'^j}{\partial q^m}dq^l dq^m, \end{aligned}$$

where labels of summed indices were changed appropriately. Thus the metric in the primed coordinate is related to the metric in the unprimed coordinate as

$$g_{lm} = g'_{ij}\frac{\partial q'^i}{\partial q^l}\frac{\partial q'^j}{\partial q^m}. \tag{9.43}$$

This type of transformation is the defining nature of a tensor object, see Appendix C; the metric is a second rank tensor. Not all indexed objects are tensors. For example the Christoffel symbols are not, see Section 9.7 and Appendix C. The quantity, $g = det\,(\boldsymbol{G})$, looks like a scalar but is not. Equation 9.43 can be cast in the form of a matrix equation:

$$\boldsymbol{G} = \frac{\boldsymbol{\partial q'}}{\boldsymbol{\partial q}} \boldsymbol{G'} \frac{\boldsymbol{\partial q'}}{\boldsymbol{\partial q}}^{\boldsymbol{T}}$$

where

$$\frac{\boldsymbol{\partial q'}}{\boldsymbol{\partial q}} \equiv \begin{pmatrix} \frac{\partial q'^1}{\partial q^1} & \frac{\partial q'^2}{\partial q^1} \\ \frac{\partial q'^1}{\partial q^2} & \frac{\partial q'^2}{\partial q^2} \end{pmatrix}. \tag{9.44}$$

and $\frac{\boldsymbol{\partial q'}}{\boldsymbol{\partial q}}^{\boldsymbol{T}}$ is the transpose. Thus

$$g = \left(det \left(\frac{\boldsymbol{\partial q'}}{\boldsymbol{\partial q}} \right) \right)^2 g'. \tag{9.45}$$

The four indexed objects R_{ijkl} are tensors of the indicated nature and thus change in different coordinate systems. The curvature on the other hand is the same in all coordinate systems. This is most easily realized by examining the Gauss map definition. Using intrinsic techniques, the curvature is given by $K_G = \frac{R_{1212}}{g}$. Since both R_{1212} and g are coordinate dependent, it is worth the effort to show the coordinate independence of the curvature:

$$\begin{aligned} \frac{R_{1212}}{g} &= \frac{\partial q'^i}{\partial q^1} \frac{\partial q'^m}{\partial q^2} \frac{\partial q'^n}{\partial q^1} \frac{\partial q'^o}{\partial q^2} \frac{R'_{imno}}{g} \\ &= \frac{\partial q'^i}{\partial q^1} \frac{\partial q'^m}{\partial q^2} \frac{\partial q'^n}{\partial q^1} \frac{\partial q'^o}{\partial q^1} \left(g'_{in} g'_{mo} - g'_{io} g'_{mn} \right) \frac{R'_{1212}}{g' g} \\ &= \left(g_{11} g_{22} - g_{12} g_{21} \right) \frac{R'_{1212}}{g' g} \\ &= \frac{R'_{1212}}{g'}. \end{aligned} \tag{9.46}$$

There is an interesting and ultimately very important counting argument here. We have shown that the important features of the surface is the curvature. It is also independent of the coordinate system used to compute it. In the Theorem Egrigium, we have shown that the we can construct the curvature from the metric and its derivatives. The metric is composed of three independent functions, g_{11}, g_{22} and $g_{12} = g_{21}$ and is coordinate dependent. The coordinate information is contained in two functions. Thus it makes sense that there is one, $3-2$, coordinate independent function, the curvature, for a two dimensional surface. In higher dimensional cases this pattern will change but the counting idea will still apply, see Section 9.7.

9.6 Examples of Two Dimensional Geometry

In the following subsections, we will examine several examples of two geometries. In all cases, we will attempt to understand the situation from an analysis of the line element alone as is consistent with the Theorema Egregium, Section 9.5. When that gets too complicated, we will resort to our understanding of the space as it is imbedded into the larger three space.

9.6.1 Euclidean Two Plane in Polar Coordinates

The simplest case has to be the Euclidean two plane. In Cartesian coordinates it has the line element $ds^2 = dx^2 + dy^2$. The metric is $g_{xx} = g_{yy} = 1$ and $g_{xy} = 0$. All the connections are zero. The curvature is zero. The geodesic equation is

$$\begin{aligned} \frac{d^2x}{ds^2} &= 0 \\ \frac{d^2y}{ds^2} &= 0 \end{aligned} \tag{9.47}$$

with the general solution

$$\begin{aligned} x &= as + b \\ y &= cs + d. \end{aligned} \tag{9.48}$$

for all a, b, c, and d.

Note that a geodesic is a more specific than just a straight line. The geodesic in Equation 9.48 is a straight line but to be a geodesic the line must also advance linearly in the length element. In other words, the system

$$\begin{aligned} x &= as^2 + b \\ y &= cs^2 + d \end{aligned} \tag{9.49}$$

is a straight line but not a geodesic.

This simplicity of the Euclidean two plane can be masked by going to a different coordinate system. A common example is the polar coordinates. In this coordinate, the line element is $ds^2 = dr^2 + r^2 d\theta^2$. In order to remove the tendency to identify the underlying space, relabel the coordinates, $y \to r, x \to \theta$, so that the line element is expressed as

$$ds^2 = y^2 dx^2 + dy^2, \tag{9.50}$$

Actually, it would be more consistent with the spirit of this section to find the line element from the coordinate transformation. In this set of labels, the coordinate transformation is

$$\begin{aligned} x &= \tan^{-1}\left[\frac{y_0}{x_0}\right] \\ y &= \sqrt{x_0^2 + y_0^2}, \end{aligned} \tag{9.51}$$

where x_0 and y_0 are the cartesian coordinates and some care must be exercised in the handling of the $\tan^{-1}$ function and the inverse,

$$\begin{aligned} x_0 &= y\cos(x) \\ y_0 &= y\sin(x). \end{aligned} \tag{9.52}$$

The transformation matrix is

$$\frac{\boldsymbol{\partial x_0}}{\boldsymbol{\partial x}} \equiv \begin{pmatrix} \frac{\partial x_0}{\partial x} & \frac{\partial y_0}{\partial x} \\ \frac{\partial x_0}{\partial y} & \frac{\partial y_0}{\partial y} \end{pmatrix} = \begin{pmatrix} -y\sin x & \cos x \\ y\cos x & \sin x \end{pmatrix}.$$

Equation 9.44 then yields $g_{xx} = y^2,\ g_{xy} = g_{yx} = 0$, and $g_{yy} = 1$, the metric in Equation 9.50.

The coordinate cover of the Euclidean two plane is shown in Figue 9.3.

In the (x, y) coordinate, the non-trivial connections are

$$\begin{aligned} \Gamma^x_{xy} &= \frac{1}{y} = \Gamma^x_{yx} \\ \Gamma^y_{xx} &= -y. \end{aligned} \tag{9.53}$$

The geodesic equations are

$$\begin{aligned} \frac{d^2x}{ds^2} + \frac{2}{y}\frac{dx}{ds}\frac{dy}{ds} &= 0 \\ \frac{d^2y}{ds^2} - y\left(\frac{dx}{ds}\right)^2 &= 0. \end{aligned} \tag{9.54}$$

From Equation 9.54, the curves generated by $\frac{dx}{ds} = 0$ or $x = c'$ and $y = a's + b'$ are geodesics for all $a',\ b'$ and c'. These are the straight lines which would

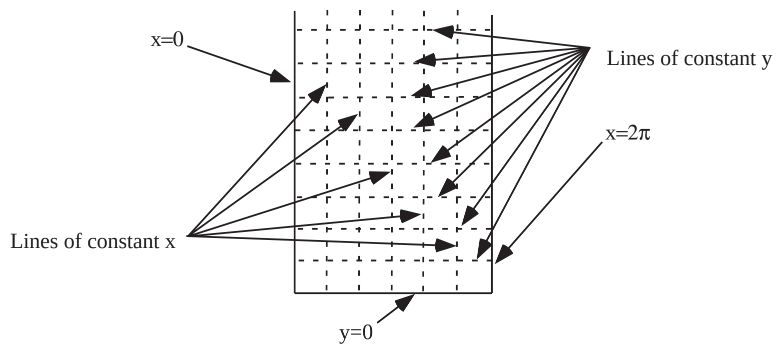

Figure 9.3: **Cover of the Plane in Polar Coordinates** The coordinate system that covers the Euclidean two plane. y is what is commonly called the radius and its range is $0 \leq y < \infty$. x is θ and its range is $0 \leq x < 2\pi$.

project through the origin identified with $\theta = \theta_0$ and the radius increasing linearly in length. These are not all the geodesics but only the subset of Equation 9.48 with $a = 0$ and $0 \leq b \leq 2\pi$ and $0 \leq c, d$. This set is a small subset of the the geodesics, Equation 9.48, since it admits only lines through the origin. That the set of curves in Equation 9.48 in the cartesian coordinate are solutions of Equation 9.54 when substituted into Equation 9.51 for all a, b, c and d can be seen by direct substitution.

It is also an interesting exercise to do some simple geometry in the Euclidean two plane in this coordinate. First, draw a fan of geodesics about some point in the Cartesian plane. Figure 9.5 is a fan of equal opening geodesics about a general point in the plane in the Cartesian plane and the image fan in the polar plane. The rays in the polar plane are given by

$$\begin{aligned} x &= \tan^{-1}\left[\frac{\sin(\phi_i)\, s + y_{0_i}}{\cos(\phi_i)\, s + x_{0_i}}\right] \\ y &= \sqrt{\left(\cos(\phi_i)\, s + x_{0_i}\right)^2 + \left(\sin(\phi_i)\, s + y_{0_i}\right)^2}, \end{aligned} \tag{9.55}$$

where ϕ_i is the opening angle to the i^{th} ray in the cartesian plane and s is the length parameter of the ray which ranges from 0 to 1. That this is the correct identification for the parameter s in the polar plot follows from the fact that we have the same line element in both coordinates or, saying the same thing, that from Equation 9.55, $y(s)^2 \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$.

The rays of the fan in Figure 9.5 are all of unit length. The same rays are shown in the two coordinate systems in Figure 9.5. Since, in this case, the rays all have unit length, the tips of rays trace out a circle of unit radius.

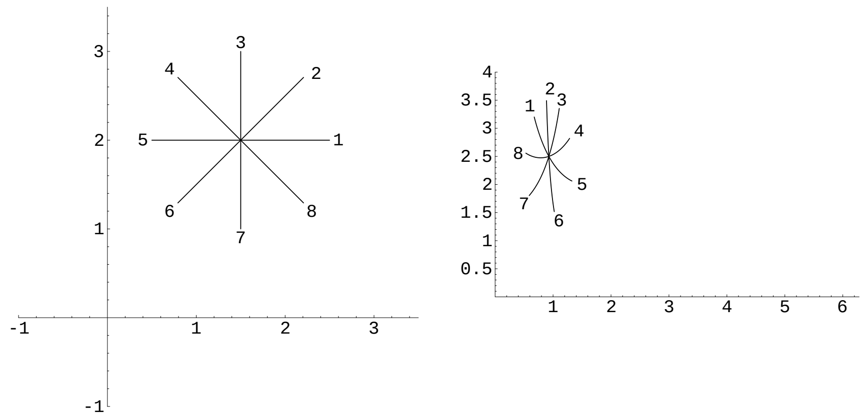

Figure 9.4: **Fan of Geodesics in the Euclidean Two Plane** On the left is a fan of geodesics starting at the point (x_{0_i}, y_{0_i}) in the Cartesian coordinates. On the right is the same fan shown in the polar coordinates plane. The corresponding rays in each case are numbered. Since the rays all have unit length, the tips also trace out the shape of the circle of radius 1, the locus of places that are unit distance from the point $\left(\tan^{-1}\left(\frac{y_{0_i}}{x_{0_i}}\right), \sqrt{x_{0_i}^2 + y_{0_i}^2}\right)$. In this case, $x_{0_i} = 1.5$ and $y_{0_i} = 2$.

In other words, we can construct a circle of radius r in the polar plane by replacing s in Equation 9.55 by r and allow ϕ_i to range continuously from zero to 2π. Using this form for the circle in the polar plot, we can see that the angles between any two intersecting geodesics is the same in the polar coordinates as in the Cartesian. Writing the line element as a function of ϕ_i for fixed r, $y(\phi_i)^2 \left(\frac{dx}{d\phi_i}\right)^2 + \left(\frac{dy}{d\phi_i}\right)^2 = r^2$. Thus the ratio of the arc length of a circle to the radius is

$$\Delta\theta \equiv \frac{\int_{arc} ds}{r} = \int_{\Delta\phi_i} d\phi_i = \Delta\phi_i \tag{9.56}$$

which, as stated above, is the angle between the corresponding geodesics in the Cartesian plane.

Thus if, as discussed in Section 9.1, we do simple geometry in the polar coordinate plane, we will get all the usual results that one obtains for a Euclidean two plane: the sum of the interior angles of a triangle is π, the ratio of the circumference to the radius of a circle is 2π, a square will close. Although these figures can look rather bizarre in the polar coordinates (draw a triangle with one vertex on the origin and examine the sum of the interior angles), this space is flat. The bizarre appearance of geometric figures is an accident of the coordinate choice. It is not due to an intrinsic property of the space. This will be a problem in some of our examples in space-time, see Section **??**, – a coordinate choice that is nice for some applications can have a non-physical appearance for certain circumstances.

9.6.2 The Unit Two Sphere in Euclidean Three Space

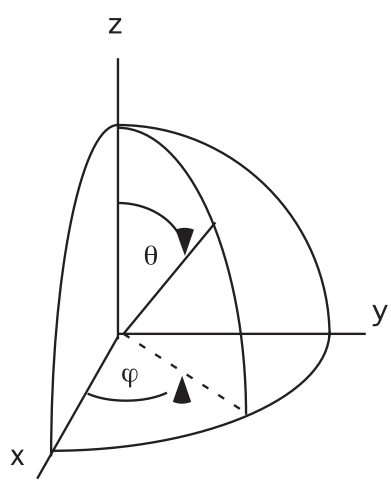

Figure 9.5: **Unit Two Sphere** The coordinates θ and ϕ defined on the unit two sphere.

Another well known example is the unit two sphere imbedded into a Euclidean three space. Using the usual spherical polar coordinates, the line element is well known to be

$$ds^2 = \sin^2(\theta)\, d\phi^2 + d\theta^2, \tag{9.57}$$

where $0 \leq \phi < 2\pi$ and $0 \leq \theta < \pi$. From this, the non-vanishing connections are

$$\begin{aligned} \Gamma^{\phi}_{\phi\theta} &= \cot(\theta) \\ \Gamma^{\theta}_{\phi\phi} &= -\cos(\theta)\sin(\theta), \end{aligned} \tag{9.58}$$

and the geodesic equations are

$$\frac{d^2\phi}{ds^2} + 2\cot(\theta)\frac{d\theta}{ds}\frac{d\phi}{ds} = 0$$

$$\frac{d^2\theta}{ds^2} - \cos(\theta)\sin(\theta)\left(\frac{d\phi}{ds}\right)^2 = 0. \tag{9.59}$$

The curves $\phi = \phi_0$, $\frac{d\phi}{ds} = 0$, and $\theta = as + \theta_0$ are geodesics lines. By requiring that s be given by Equation 9.57, $a = 1$. Of course, these are the well known lines of longitude passing through the point (ϕ_0, θ_0). Thus with suitable evolution in angle, θ, all lines of longitude are geodesics. These lines all have length 2π. For $\theta = \theta_0$, $\frac{d\theta}{ds} = 0$, the only geodesics that are easy to see in Equation 9.59, are at $\theta_0 = 0, \pi, \frac{\pi}{2}$. From the line element, Equation 9.57, two of these, $\theta_0 = 0$ and π, have zero length and are thus not lines at all. The remaining case is the equatorial line and, with the proper evolution of ϕ, $\phi = s + \phi_0$, is a geodesic. Again, the length of this geodesic is 2π.

Of course, these are not all of the geodesics. There is an infinity of them through each point in the two plane. Instead of solving the geodesic equations, Equations 9.59, we will us the extrinsic properties of the sphere imbedded in a Euclidean three space. Given the results of the Theorema Egregium, Section 9.5, we should not resort to such subterfuge but it is clear from the form of the geodesic equations that general solutions are not easy to obtain. On the other hand, the sphere is so symmetric that we can rotate about any axis through the origin and generate additional geodesics from the simple ones that we have. For example, rotating the sphere about the x axis by an angle α moves the equatorial geodesic to a new orientation. This is the well known result that geodesics on a sphere are great circle paths. Thus for any point, (ϕ_0, θ_0), we can find a pair of orthogonal geodesics passing through the point by using a line of longitude, $(\phi = \phi_0, \theta = s + \theta_0)$ and the equatorial line, $\left(\phi = s + \phi_0, \theta = \frac{\pi}{2}\right)$ with $\phi_0 = \frac{\pi}{2}$ rotated about the x axis by an angle of $\frac{\pi}{2} - \theta_0$ and then rotating this sphere about the z axis by the angle $\phi_0 - \frac{\pi}{2}$. This produces the geodesic

$$\begin{aligned}
\phi &= \tan^{-1}\left\{\frac{\cos\phi_0 \sin s + \sin\phi_0 \sin\theta_0 \cos s}{-\sin\phi_0 \sin s + \cos\phi_0 \sin\theta_0 \cos s}\right\} \\
\theta &= \tan^{-1}\left\{\frac{\sqrt{\sin^2 s + \sin^2\theta_0 \cos^2 s}}{\cos\theta_0 \cos s}\right\}.
\end{aligned} \tag{9.60}$$

That Equations 9.60 are geodesics can be confirmed by direct substitution into the geodesics equations, Equation 9.59. Again, of course, this development of the geodesic equations is a violation of the spirit of the the Theorema Egregium, Section 9.5, since we are using the rotational symmetry of the sphere which at this point can only be inferred extrinsically. We will

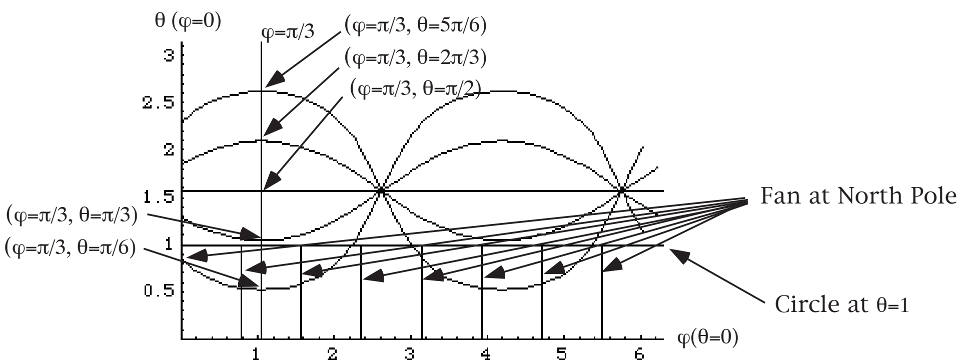

Figure 9.6: **Geodesics, Fan, and Circle of Radius One on the Unit Sphere in the** (ϕ, θ) **Plane** A series of orthogonal geodesics along the $\phi = \frac{\pi}{3}$ geodesic. This sequence of geodesics shows important properties of positively curved non-Euclidean spaces. Parallel geodesics that are locally parallel at a point curve toward each other. In this case, these parallel geodesics actually meet. In addition, the figure show a fan emerging from the north pole, $(\phi = 0, \theta = 0)$ with equal opening angles of $\frac{\pi}{4}$ between elements and a circle centered on the origin of radius one.

need some measure of symmetry in a general coordinate or on the general manifold to use a trick like this to generate geodesics. How this can be done will be addressed later later in Section 9.7. For now, we can proceed using these geodesics to do some simple geometry. In a $(\phi\ \theta)$ plot, we can show the nature of the geodesics for some ϕ_0, actually $\phi_0 = \frac{\pi}{3}$, and several θ see Figure 9.6. Along the line of longitude geodesic, the orthogonal geodesics are parallel. Yet all these geodesics move toward each other and, in this case, even meet. This is the signature of the positively curved non-Euclidean two space. This is a direct consequence of the abandonment of Euclid's axiom of parallels – For a line and an external point, there is one and only one line through the external point parallel to the original line and that line when extended never intersects the original line . For example for our case, given any of the geodesics, for example the simplest one, $\theta = \frac{\pi}{2}$, and an external point, (ϕ_0, θ_0), the geodesic given by Equation 9.60 produces the unique geodesic that is locally parallel to the original geodesic. As this line through (ϕ_0, θ_0) is extended along the geodesic it moves closer to the original line and ultimately intersects it. The other feature of this homogeneous positively curved non-Euclidean space is that since they are generated by rotation of special geodesics, all these geodesics have the same length and this length is finite and equals 2π. Since the geodesic equations are invariant under changes of coordinate choices, these properties are independent of the coordinate choice and reflect the intrinsic geometry of the manifold.

Figure 9.6, also shows a fan of equal length rays, length equal to one, emerging from the north pole in the imbedded unit sphere and the enclosing circle. Again, we can use the homogeneity of the imbedded sphere to draw conclusions about a circle and fan drawn anywhere on the sphere and this circle, although it does not look natural in the (ϕ, θ) plane, is particularly simple to analyze. If we now make a fan with arbitrary ray length, r, and do an analysis similar to that of the flat plane example in Equation 9.56,

$$\Delta\theta \overset{?}{\equiv} \frac{\int_{arc} ds}{r} = \frac{1}{r}\int_{\Delta\phi_i} \sin(r)\, d\phi_i = \frac{\sin(r)}{r}\Delta\phi_i, \tag{9.61}$$

where $\Delta\phi_i$ is the difference in the labels of the adjacent rays. In the case of this definition, the angle would depend on the size of the radius used. Again, this is a characteristic feature of non-Euclidean spaces. The correct definition of angle for any space is thus

$$\Delta\theta \equiv \lim_{r\to 0} \frac{\int_{arc} ds}{r} \tag{9.62}$$

which for our case, yields a result consistent with our intuition. This definition will always work since, for all non-singular regions, the geometry of any Riemann manifold is Euclidean for a sufficiently small region. The idea of sufficiently small will be clarified when we discuss manifolds more generally in Section 9.7.

9.6.3 The Two Torus

As a third example, consider the case of the torus discussed in Section 9.3. Using the coordinates of that section, Equations 9.2, the line element is

$$ds^2 = [R_2 - R_1 \sin\theta]^2\, d\phi^2 + R_1^2 d\theta^2. \tag{9.63}$$

The non-zero connections are

$$\begin{aligned} \Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} &= -\frac{R_1 \cos(\theta)}{R_2 - R_1 \sin(\theta)} \\ \Gamma^{\theta}_{\phi\phi} &= \frac{\cos(\theta)\,[R_2 - R_1 \sin(\theta)]}{R_1}. \end{aligned} \tag{9.64}$$

Using these connections, the Riemann tensor is, $R_{1212} = -R_1\left[R_2 - R_1 \sin(\theta)\right]$, and curvature is $K_G = \frac{\sin(\theta)}{R_1[R_2 - R_1\sin(\theta)]}$ which agrees with Equation 9.3 which we derived using the Gauss map.

Again using the connections of Equation 9.64, the geodesic equations are

$$\begin{aligned}\frac{d^2\phi}{ds^2} - 2\frac{R_1\cos(\theta)}{R_2 - R_1\sin(\theta)}\frac{d\theta}{ds}\frac{d\phi}{ds} &= 0\\ \frac{d^2\theta}{ds^2} + \frac{\cos(\theta)\left[R_2 - R_1\sin(\theta)\right]}{R_1}\left(\frac{d\phi}{ds}\right)^2 &= 0.\end{aligned} \tag{9.65}$$

The case $\frac{d\phi}{ds} = 0$ or $\phi = a$ has a solution for θ of $\theta = bs + c$. In other words the lines of constant ϕ are geodesics if the θ advance is appropriate. Requiring that s be the path length parameter, $R_1^2\left(\frac{d\theta}{ds}\right)^2 = 1$, makes $b = \frac{1}{R_1}$. The more interesting case is lines of constant θ which would require $\phi = bs + c$. These are geodesics only for $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

The exact solution of the geodesic equations, Equations 9.65, are not obvious and instead of relying on an extrinsic solution, it will be sufficient for our purposes to find an approximate solution. Given the differential form of the geodesic equations, it is reasonable to assume that from a point (ϕ_0, θ_0), the geodesics can be expressed as a smooth function of the path length. Using the dimensionless expansion parameter $\frac{s}{R_2}$

$$\begin{aligned}\phi(s) &= \phi_0 + a\frac{s}{R_2} + b\left(\frac{s}{R_2}\right)^2 + O\left(\left(\frac{s}{R_2}\right)^3\right)\\ \theta(s) &= \theta_0 + c\frac{s}{R_2} + d\left(\frac{s}{R_2}\right)^2 + O\left(\left(\frac{s}{R_2}\right)^3\right),\end{aligned} \tag{9.66}$$

There are four constants, a through d, and at $s = 0$ four conditions:

1. That s is the path length or from Equation 9.63, $\left[R_2 - R_1\sin\theta\right]^2\left(\frac{d\phi}{ds}\right)^2 + R_1^2\left(\frac{d\theta}{ds}\right)^2 = 1$.

2. The angle α relative to the ϕ axis of the geodesic, $\frac{\frac{d\theta}{ds}}{\frac{d\phi}{ds}} = \tan(\alpha)$.

3. The two geodesic equations, Equation 9.65.

We can most easily find an orthogonal pair of geodesics using the already known one, $\phi = \phi_0$ and $\theta = \theta_0 + \frac{1}{R_1}s$ and one with emergent angle $\alpha = 0$. These will be the closest that we can come to the constant ϕ, constant θ lines as geodesics. At $s = 0$, the second condition above requires that $\frac{d\theta}{ds}|_{s=0} = 0$ or $c = 0$. Since $\frac{d\theta}{ds}|_{s=0} = 0$, the first condition requires that $a = \frac{1}{[R_2 - R_1 \sin\theta]}$ or $\phi(s) = \phi_0 + \frac{s}{[R_2 - R_1 \sin\theta]} + bs^2 + O(s^3)$. Plugging this into the ϕ geodesic equation at $s = 0$, requires that $b = 0$. The θ geodesic similarly requires requires $d = -\frac{\cos(\theta_0)}{2R_1[R_2 - R_1 \sin(\theta_0)]}$. Thus the two parametric equations for the second geodesic are

$$\begin{aligned}
\phi(s) &= \phi_0 + \frac{s}{[R_2 - R_1 \sin(\theta_0)]} + O\left(\left(\frac{s}{R_2}\right)^3\right) \\
\theta(s) &= \theta_0 - \frac{\cos(\theta_0)}{2R_1[R_2 - R_1 \sin(\theta_0)]} s^2 + O\left(\left(\frac{s}{R_2}\right)^3\right) \qquad (9.67)
\end{aligned}$$

Using the first of these geodesic equations to solve for s,

$$s \approx [R_2 - R_1 \sin(\theta)](\phi - \phi_0),$$

it is easy to find the shape of these geodesics,

$$\theta - \theta_0 \approx -\frac{\cos(\theta_0)[R_2 - R_1 \sin(\theta_0)]}{2R_1}(\phi - \phi_0)^2. \qquad (9.68)$$

This is an interesting result. Figure 9.7 shows a set of these geodesics along the geodesic $\phi = \frac{\pi}{3}$. From Equation 9.68, these geodesics are parabolas with concavity up when the initial θ is between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ and negative otherwise. The concavity is zero when θ_0 is $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. This produces a pattern of initially parallel geodesics that are approaching each other in regions of positive curvature and receding from each other in regions of negative curvature. This approach and recession pattern is, of course, the signature of the sign of the curvature. It was this pattern that will allow us to identify the sign of the curvature in the space-time two planes for the free fall bodies in the vicinity of the earth, see Section 8.5.2.

Figure 9.7, also shows two fans one located in the region of positive curvature and one in the region of negative curvature. The rays of the fans are all of unit length and thus the tips sketch out the shape of the circle in the (ϕ, θ) plane.

9.6.4 The Accelerated Reference Frame Revisited

In Section 6.5, we developed a coordinate system based on the construction by an accelerated observer. The metric is given in Equation 6.36 or

$$g_{\alpha\beta} = \begin{pmatrix} e^{\left(\frac{2gx}{c^2}\right)} & 0 \\ 0 & -e^{\left(\frac{2gx}{c^2}\right)} \end{pmatrix}. \qquad (9.69)$$

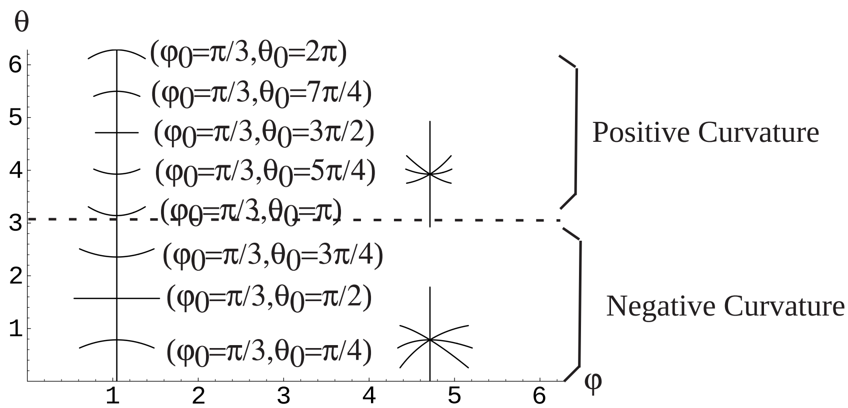

Figure 9.7: **Geodesics and Fans on the Two Torus in the** (ϕ, θ) **Plane** A series of orthogonal geodesics on the two torus along the $\phi = \frac{\pi}{3}$ geodesic. This sequence of geodesics shows important differences between positively curved and negatively curved non-Euclidean spaces. Parallel geodesics that are locally parallel at a point curve toward each other when the curvature is positive and away when the curvature is negative. In addition, the figure shows two fans with rays of unit length emerging from points located in the positively curved region and in the negatively curved region. These fans are centered on $\left(\phi = \frac{3\pi}{2}, \theta = \frac{5\pi}{4}\right)$ and on $\left(\phi = \frac{3\pi}{2}, \theta = \frac{\pi}{4}\right)$ respectively.

The inverse metric is

$$(g_{\alpha\beta})^{-1} = g^{\alpha\beta} = \begin{pmatrix} e^{\left(\frac{-2gx}{c^2}\right)} & 0 \\ 0 & -e^{\left(\frac{-2gx}{c^2}\right)} \end{pmatrix} \tag{9.70}$$

9.6.5 The Hyperbolic Space-Time Two Surface

Let's conclude with a case in space-time. Our underlying space is the usual Minkowski space with 2 spacelike directions, x and y, and one timelike direction,t, see Section 4.3. If we take the invariant curve for Lorentz transformation in the $x - t$ plane, see Section 4.5, and generate a surface of revolution about the t axis, we will generate a non-trivial two surface that has both time-like and space-like trajectories. This will provide us with an interesting non-trivial two dimensional space-time. Using as coordinates, the proper time along the original invariant curve and the rotation angle, the metric, see Section **??** is

There is an alternative method for generating hyperbolic surfaces and it points out an important feature of these surfaces. The hyperbolic two surface embedded in a three space is the envelope of the straight lines with a given inclination and these are the only straight lines in the embedding three space that are in the two space. Any other curve in the surface is curved in the three space. Figure 9.8 shows the surface as generated by these straight lines. When I was young, many trash receptacles were constructed in this fashion.

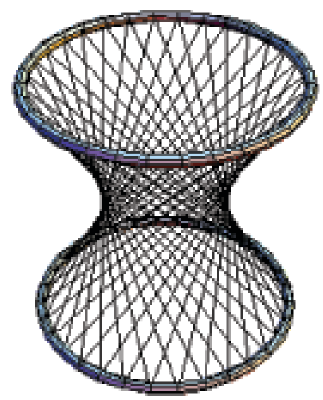

Figure 9.8: **Hyperboloid**

9.7 Geometry in Four or More Dimensions

9.8 Some notation and nomenclature:

The metric is g_{ij} is called the first fundamental form and is defined as

$$g_{\mu\nu} \equiv \frac{\partial x^{\gamma}}{\partial q^{\mu}} \frac{\partial x_{\gamma}}{\partial q^{\nu}} \tag{9.71}$$

The second fundamental form is $b_{\mu\nu}$ is

$$b_{\mu\nu} = \frac{\partial^2 x^{\gamma}}{\partial q^{\mu} \partial q^{\nu}} n_{\gamma} \tag{9.72}$$

The mixed Riemann curvature tensor is

$$R^{\mu}_{\nu\gamma\rho} = \frac{\partial \Gamma^{\mu}_{\nu\rho}}{\partial q^{\gamma}} - \frac{\partial \Gamma^{\mu}_{\nu\gamma}}{\partial q^{\rho}} + \Gamma^{\eta}_{\nu\rho}\Gamma^{\mu}_{\eta\gamma} - \Gamma^{\eta}_{\nu\gamma}\Gamma^{\mu}_{\eta\rho} \tag{9.73}$$

where

$$\Gamma^{\mu}_{\nu\gamma} = g^{\mu\eta}\Gamma_{\nu\gamma,\eta} \tag{9.74}$$

where

$$\Gamma_{\nu\gamma,\eta} = \frac{1}{2}\left\{\frac{\partial g_{\eta\gamma}}{\partial q^{\nu}} + \frac{\partial g_{\eta\nu}}{\partial q^{\gamma}} - \frac{\partial g_{\nu\gamma}}{\partial q^{\eta}}\right\} \tag{9.75}$$

and the Riemann curvature tensor is

$$R_{\mu\nu\rho\gamma} = R^{\eta}_{\nu\rho\gamma}g_{\eta\mu} \tag{9.76}$$

The Ricci tensor is

$$R_{\mu\nu} \equiv R^{\rho}_{\mu\rho\nu} \tag{9.77}$$

The symmetries of the Riemann curvature tensor are:

$$R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu} \tag{9.78}$$

$$R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = R_{\mu\lambda\kappa\nu} \tag{9.79}$$

The cyclicity condition:

$$R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0 \tag{9.80}$$

9.9 Coordinate Labels in General Relativity

9.10 Distances and Time Intervals

At a fixed spatial coordinate, the time interval that lapses for an increment in the local coordinate time is given by

$$c^2 d\tau^2 = g_{00}c^2 dt^2. \tag{9.81}$$

Thus the lapsed time is

$$\tau(t) = \frac{1}{c}\int_{x_i^0}^{x^0} \sqrt{g_{00}}dx'^0 = \int_{t_0}^{t} \sqrt{g_{00}}dt' \tag{9.82}$$

Note that this implies that $g_{00} > 0$. This is not the same condition that the metric must have principal eigenvalues such the there is one positive and three negative. That is a condition that the metric be that of a real gravitational field – a metric for space-time. The condition $g_{00} > 0$ is one that says that the system of reference could not be realized by material bodies. An example is the coordinate system rotating at a rate Ω. Using cylindrical coordinates,

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \tag{9.83}$$

and the coordinate transformation

$$\begin{aligned} r &= r' \\ \phi &= \phi' - \Omega t' \\ z &= z' \\ t &= t', \end{aligned} \tag{9.84}$$

the interval becomes

$$c^2 d\tau^2 = \left(c^2 - r'^2\Omega^2\right) dt'^2 - dr'^2 - r'^2 d\phi'^2 - dz'^2 + 2r'^2\Omega d\phi' dt' \tag{9.85}$$

The condition $g_{00} > 0$ is violated when a rigid merry-go-round could be utilized as the basis for the coordinate transformation. The metric condition is the condition that the merry-go-round could rotate at the rate Ω and be extended to distance $D > \frac{c}{\Omega}$.

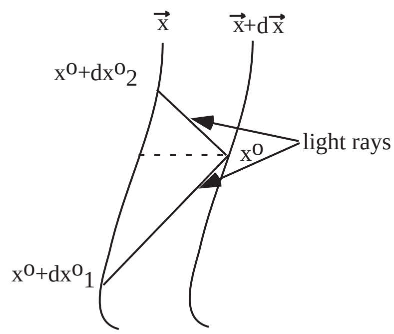

Figure 9.9: **Distance in a General Relativity** Given a metric on the coordinates, the spatial distance between two nearby coordinate points is determined by the radar method, see Section 3.1, as the speed of light times the time between transmission and response to the nearby point divided by two.

Spatial distances must be measured by the radar method, see Section 3.1. For two places infinitesimally close, a light ray traveling from one coordinate place to another and back is as shown in Figure 9.9. The trajectory of the light ray is given by

$$c^2 d\tau^2 = 0 = g_{00} dx^{0^2} + 2g_{0a} dx^0 dx^a + g_{ab} dx^a dx^b. \tag{9.86}$$

This is a quadratic for the coordinate time increment and thus there are the two solutions represented in Figure 9.9.

$$dx_1^0 = -\frac{1}{g_{00}}\left\{g_{0a}dx^a - \sqrt{(g_{0a}g_{0b} - g_{00}g_{ab})\,dx^a dx^b}\right\} \tag{9.87}$$

and

$$dx_2^0 = -\frac{1}{g_{00}}\left\{g_{0a}dx^a + \sqrt{(g_{0a}g_{0b} - g_{00}g_{ab})\,dx^a dx^b}\right\} \tag{9.88}$$

The difference in the coordinate time is

$$dx_2^0 - dx_1^0 = \frac{2}{g_{00}}\sqrt{(g_{0a}g_{0b} - g_{00}g_{ab})\,dx^a dx^b}, \tag{9.89}$$

and, thus, the spatial distances are

$$dl^2 = \left(g_{ab} - \frac{g_{0a}g_{0b}}{g_{00}}\right) dx^a dx^b. \tag{9.90}$$

Thus, the three space metric, γ_{ab}, is

$$\gamma_{ab} = \left(g_{ab} - \frac{g_{0a}g_{0b}}{g_{00}}\right). \tag{9.91}$$

The inverse of this three space metric, γ_{ab}, is g^{ab}, and it is three space part of original four space metric, $g_{\mu\nu}$. This comes about since $g^{\mu\nu}g_{\nu\alpha} = \delta^\mu_\alpha$ and, therefore, $g^{ab}g_{bc} + g^{a0}g_{0c} = \delta^a_c$ and $g^{ab}g_{b0} + g^{a0}g_{00} = 0$. Solving for $ga0$ from the second equation and substituting into the first, we have $g^{ab}\gamma_{bc} = \delta^a_c$.

Once the line element has been determined in any given coordinate, we can address the important issue of simultaneity. Obviously, in a general coordinate system, the surfaces of constant coordinate time will not necessarily and, most likely, will not be surfaces of simultaneity. Like the problems of time and distance above there is really no global construction of the surfaces of simultaneity. Again, we use the radar method, see Section 3.1, to establish simultaneity at nearby spatial coordinates. In that case, the event at the trajectory associated with the position $\vec{x}$ that is simultaneous with the event labeled x^0 on the trajectory with position $\vec{x} + d\vec{x}$ is $x^0 + \frac{dx_1^0 + dx_2^0}{2}$. From Equation 9.87 and Equation 9.88,

$$\Delta x^0 = -\frac{g_{0a}dx^a}{g_{00}}. \tag{9.92}$$

We have already seen an application of this result in Section 6.5. Had we stuck with the classic confederate scheme for coordinates using the local clocks, Equation 6.20, the lines of constant τ' would not have been lines of simultaneity. This is indicated by the line element, which, from Equation 6.20, is

$$c^2 d\tau_{proper}^2 = c^2 d\tau'^2 - 2\frac{\left(\frac{g\tau'}{c}\right)}{\left(1+\frac{gh}{c^2}\right)} c d\tau' dh - \left(1 - \frac{\left(\frac{g\tau'}{c}\right)^2}{\left(1+\frac{gh}{c^2}\right)^2}\right) dh^2. \quad (9.93)$$

Using this line element and Equation 9.92, we have that the event labeled by (h, τ') is simultaneous with the nearby event $\left(h + \Delta h, \tau' + \frac{\frac{g\tau'}{c}}{1+\frac{gh}{c^2}}\Delta h\right)$. This is consistent with our previous result that the time coordinate, $\tau \equiv \frac{\tau'}{1+\frac{gh}{c^2}}$, generates lines of simultaneity for the accelerated coordinate system of Section 6.5 in that the line between these events has the same slope, or in this case hangle, as the line from the magic point $\left(-\frac{c^2}{g}, 0\right)$ to the event (h, τ') and, obviously, both lines pass through that event.

As in the previous cases, this is a local rule and its use for more than infinitesimal separations is somewhat problematic. In the case above of the accelerated system, there is no problem with extending the result to large separations. This will not be the case generally. Consider the case of the merry-go-round. From the line element, Equation 9.85, the shift in coordinate time to synchronize clocks for a shift in angle, ϕ' at fixed r', z', and t' is $\Delta x^0 = -\int_{\phi_0'}^{\phi'} \frac{r'^2\Omega}{c^2 - r'^2\Omega} d\phi''$. A closed curve then yields and inconsistent result that the clock at at fixed r', ϕ_0', z', and t' has to be shifted by the amount $-\frac{2\pi r'^2\Omega}{c^2 - r'^2\Omega}$ to be brought into synchronization with itself. Of course, in this coordinate system, the g_{hh} metric element is not the appropriate measure of the spatial length of the coordinate change Δh. It is instead, Equation 9.91,

$$\gamma_{hh} = \left(g_{hh} - \frac{g_{\tau' h} g_{\tau' h}}{g_{\tau'\tau'}}\right) = -1. \quad (9.94)$$

While we are dealing with these issues of the import of the line element, note how nicely the event horizon for the accelerated system appears in the line element, Equation 9.93. At fixed τ', changes in the coordinate h produce no change in length at the places that

$$\left(1 - \frac{\left(\frac{g\tau'}{c}\right)^2}{\left(1+\frac{gh}{c^2}\right)^2}\right) = 0. \quad (9.95)$$

This is the events

$$\frac{c^2}{g} + h_0 = \pm c\tau_0', \tag{9.96}$$

which are the light-like asymptotes through the magic point.

9.11 Einstein Equations

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -\frac{8\pi \mathrm{G}}{c^4}T^{\mu\nu} \tag{9.97}$$